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Entropy of Intuitionistic Fuzzy-Dynamical Systems on *MV*-algebras

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Abstract: In this paper, intuitionistic fuzzy-dynamical systems (IF-dynamical systems) on an *MV*-algebra are introduced. Also the entropy of IF-dynamical systems is introduced and studied. Several related results are proved.

Keywords: *MV*-algebra, entropy, IF-dynamical system.

1. Introduction

The classical logic can be described by the Boolean algebra and the quantum logic can be described by orthomodular lattices. In the study of quantum logics, *MV*-algebras play an analogous role to that of Boolean algebras in classics logic. Now, we recall some elementary notions and conclusions about *MV*-algebras.

An algebra $(M,0,1,',\oplus,\odot)$ is said to be an *MV*-algebra iff it satisfies the following conditions:

(1) $a\oplus b=b\oplus a$,
(2) $(a\oplus b)\oplus c=a\oplus(b\oplus c)$,
(3) $a\oplus 0=a$,
(4) $a\oplus 1=1$,
(5) $0'=1$ and $1'=0$,
(6) $a\odot b=(a'\oplus b')'$,
(7) $a\oplus(a\oplus b')'=b\oplus(b\oplus a')'$,

We note that (7) ensures $(a')'=a$ for all $a\in M$, i.e., the complementation $':M\longrightarrow M$ is an involutive mapping. In addition, by the De Morgan law (6), the operation $\odot$ is associative and symmetric, with the neutral element 1 and annihilator 0. Of course, by the Mundici theorem [10], any *MV*-algebra can be represented by a commutative *L*-group. Recall that a commutative *L*-group is an algebraic system $(G,+,\leqslant)$, where $(G,+)$ is a commutative group, $(G,\leqslant)$ is a partially

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ordered set, being a lattice and $a \leqslant b$ implies $a+c \leqslant b+c$. Let $(G,+,\leqslant)$ be a commutative L-group, 0 be a neutral element of $(G,+)$ and $u \in G, u > 0$. On the interval $< 0,u >$, define the following operations:

$$a \oplus b = (a+b) \wedge u,$$

$$a \odot b = (a+b-u) \vee 0,$$

$$a' = u-a.$$

Then, $MG = (< 0,u >, \oplus, \odot, ', u, 0)$ becomes an MV-algebra. The Mundici theorem states that to any MV-algebra, there exists a commutative L-group G with a strong unit u (i.e., for any $a \in G$, there exists $n \in N$ such that $a \leqslant nu$) such that (G,u) is unique up to isomorphism and $M \simeq MG$. A product MV-algebra can be defined as an algebraic system

$$(M, \oplus, \odot, \cdot, ', u, 0),$$

where $(M, \oplus, \odot, ', u, 0)$ is an MV-algebra and $\cdot$ is a binary operation, satisfying the following conditions:

(i) $u \cdot u = u$,
(ii) The operation $\cdot$ is commutative and associative,
(iii) If $a+b \leqslant u$, then $c \cdot a + c \cdot b \leqslant u$ and $c \cdot (a+b) = c \cdot a + c \cdot b$ for any $c \in M$,
(iv) If $a_n \searrow 0, b_n \searrow 0$, then $a_n \cdot b_n \searrow 0$.

By using the notion of a state on an MV-algebra, one can introduce the entropy of partitions in a MV-algebra, which is a useful tool in the study of dynamical systems and their isomorphism. If two dynamical systems are isomorphic, they have the same entropy. Therefore, systems with different entropies cannot be isomorphic. The concept of entropy plays a major role in thermodynamics and statistical mechanics. It serves to describe the amount of uncertainty in physical systems.

The notion of the entropy of dynamical systems on an MV-algebra M is introduced in [13, 14, 16]. If M is a product MV-algebra, then by a dynamical system in M, we shall understand a triple (M,m,U), where $m : M \longrightarrow [0,1]$ is a state on M and $U : M \longrightarrow M$ is a mapping satisfying the following conditions:

(i) If $a+b \leq u$ then $U(a)+U(b) \leq u$ and $U(a+b) = U(a)+U(b)$,
(ii) $U(a \cdot b) = U(a) \cdot U(b)$,
(iii) $U(u) = u$,
(iv) $m(U(a)) = m(a)$ for any $a \in M$.

A partition of unity u in M is a set $A = \{a_1, ..., a_n\}$ of elements of M such that $a_1 + ... + a_n = u$. Then, the entropy of partition A is defined by

$$H(A) = \sum_{i=1}^{n} f(m(a_i)), \tag{1}$$

where $f(x) = -x \log x$, if $x > 0$, $f(0) = 0$ and the conditional entropy is defined by

$$H(A|B) = \sum_{i=1}^{n} \sum_{j=1}^{m} m(b_j) f(\frac{m(a_i \cdot b_j)}{m(b_j)}), \tag{2}$$

omitting the $j-$terms when $m(b_j) = 0$. Also

$$h(U, A) = \lim_{n \longrightarrow \infty} \frac{1}{n} H(\bigvee_{i=0}^{n-1} U^i(A)).$$

Further, the entropy of dynamical system is defined by

$$h(U) = \sup\{h(U, A); A \text{ is a partition}\}.$$

The theory of fuzzy sets introduced by [18] has showed meaningful applications in many fields of study. The idea of fuzzy set is welcome, because it handles uncertainty and vagueness, which a cantorian set could not address. In fuzzy set theory, the membership of an element in a fuzzy set is a single value between zero and one. However, in reality, it may not always be true that the degree of non-membership of an element in a fuzzy set is equal to 1 minus the membership degree, because there may be some hesitation degree. Therefore, a generalization of fuzzy sets was introduced by K. Atanassov [2] as intuitionistic fuzzy sets (IFS). IFSs incorporate the degree of hesitation, called hesitation margin. The notion of defining intuitionistic fuzzy sets as generalized fuzzy sets is quite interesting and useful in many application areas. The knowledge and semantic representation of intuitionistic fuzzy sets becomes more meaningful, resourceful and applicable, since it includes the degree of belongingness, degree of non-belongingness and the hesitation margin [1].

An IF-set (IFS) A on a space Ω is understood as a couple (μ_A, ν_A), where $\mu_A : \Omega \longrightarrow \langle 0, 1\rangle, \nu_A : \Omega \longrightarrow \langle 0, 1\rangle$ such that $\mu_A(\omega) + \nu_A(\omega) \leqslant 1$ for each $\omega \in \Omega$. The function μ_A is called the membership function, the function ν_A is called the non membership function. An IFS $A = (\mu_A, \nu_A)$ is called IF-event, if μ_A, ν_A are measurable with respect to a σ-algebra of subsets of Ω. In [9], K. Lendelova and J. Petrovicova introduced representation of IF-probability on MV-algebras.

In the present paper, we shall use a more general situation. Instead of the set of all measurable functions with values in $\langle 0, 1\rangle$, we consider any MV-algebra M and instead of IF-events, we consider the family $\mathcal{F}$ of all couples (a, b) of elements of M such that $a + b \leqslant u$. We introduce the entropy of an IF-dynamical system on an MV-algebra. In Section 2, IF-dynamical systems on a product MV-algebra M are introduced. In Section 3, notions of an IF-partition A of an MV-algebra M, common refinements of IF-partitions, the entropy $H_\alpha(A)$ of A, the conditional entropy $H_\alpha(A|B)$, where A and B are IF-partitions on a product MV-algebra M, are introduced and studied. In the subsequent section, we introduce and study the notion of the entropy of an IF-dynamical system on an arbitrary MV-algebra.

2. IF-dynamical Systems on Product MV-Algebras

Let M be an MV-algebra. Then, we consider the family $\mathcal{F}$

$$\mathcal{F} = \{(a,b); a+b \leqslant u, a,b \in M\},$$

as a partially ordered set with ordering

$$A_1 = (a_1,b_1) \leqslant (a_2,b_2) = A_2 \Longleftrightarrow a_1 \leqslant a_2, b_1 \geqslant b_2.$$

With respect to this ordering, $\mathcal{F}$ is a lattice with the operations $(a_1,b_1) \vee (a_2,b_2) = (a_1 \vee a_2, b_1 \wedge b_2)$, $(a_1,b_1) \wedge (a_2,b_2) = (a_1 \wedge a_2, b_1 \vee b_2)$, the least element $(0,u)$ and the greatest element $(u,0)$.

The following two operations $\oplus, \odot$ have been introduced on $\mathcal{F}$:

$$A_1 \oplus A_2 = (a_1,b_1) \oplus (a_2,b_2) = (a_1 \oplus a_2, b_1 \odot b_2),$$

$$A_1 \odot A_2 = (a_1,b_1) \odot (a_2,b_2) = (a_1 \odot a_2, b_1 \oplus b_2).$$

Definition 1. By an IF-state on $\mathcal{F}$, we understand any function $s : \mathcal{F} \longrightarrow [0,1]$ satisfying the following properties:

(1) $s((u,0)) = 1, s((0,u)) = 0$,
(2) $s(A \oplus B) + s(A \odot B) = s(A) + s(B)$ for any $A, B \in \mathcal{F}$,
(3) if $A_n \nearrow A$, then $s(A_n) \nearrow s(A)$.

It has been shown in [8], that, to any IF-state $s : \mathcal{F} \longrightarrow [0,1]$, there exists $\alpha \in [0,1]$ and a state $m_\alpha : M \longrightarrow [0,1]$ such that

$$s_\alpha((a,b)) = s((a,b)) = \alpha m_\alpha(a) + (1-\alpha)(1-m_\alpha(b)).$$

Definition 2. Let M be a product MV-algebra and $\mathcal{F} = \{(a,b); a+b \leqslant u, a,b \in M\}$. Define a partial binary operation $\boxplus$ and a binary operation $\boxdot$ as follows: for any (a_1,b_1), $(a_2,b_2) \in \mathcal{F}$.

$$(a_1,b_1) \boxplus (a_2,b_2) = (a_1+a_2, b_1+b_2-u), \tag{3}$$

whenever $a_1 + a_2 \leqslant u$ and $0 \leqslant b_1 + b_2 - u \leqslant u$, and

$$\begin{aligned}(a_1, b_1) \boxdot (a_2, b_2) &= (a_1 \cdot a_2, u - (u - b_1) \cdot (u - b_2)) \\ &= (a_1 \cdot a_2, -(-b_1 \cdot u) - (-b_2 \cdot u) - (-b_1 \cdot -b_2)).\end{aligned}$$

Lemma 1. For each $A, B, C \in \mathcal{F}$, the operations $\boxplus$ and $\boxdot$ have the following properties:

(1) If $A \boxplus B$ is defined, then $B \boxplus A$ is defined and $A \boxplus B = B \boxplus A$,
(2) $A \boxdot B = B \boxdot A$,
(3) If $A \boxplus B, (A \boxplus B) \boxplus C$ are defined, then $B \boxplus C, A \boxplus (B \boxplus C)$ are defined and $(A \boxplus B) \boxplus C = A \boxplus (B \boxplus C)$,
(4) $(A \boxdot B) \boxdot C = A \boxdot (B \boxdot C)$,
(5) If $B \boxplus C$ is defined, then $(A \boxdot B) \boxplus (A \boxdot C)$ is defined and $A \boxdot (B \boxplus C) = (A \boxdot B) \boxplus (A \boxdot C)$.

Proof. We only prove (5). Let $A = (a, b)$, $B = (c, d)$ and $C = (e, f)$. Since $B \boxplus C$ is defined, it follows that

$$c + e \leq u, \quad 0 \leq d + f - u \leq u.$$

Therefore,

$$a \cdot c + a \cdot e \leq u,$$

and since $0 \leq d + f - u \leq u$, we have

$$\begin{aligned}0 \geq -d - f + u \geq -u &\Longrightarrow u \geq (u - d) + (u - f) \geq 0 \\ &\Longrightarrow u \geq (u - b) \cdot (u - d) + (u - b) \cdot (u - f) \geq 0 \\ &\Longrightarrow 0 \leq u - (u - b) \cdot (u - d) - (u - b) \cdot (u - f) \leq u.\end{aligned}$$

Therefore, $(A \boxdot B) \boxplus (A \boxdot C)$ is defined. Also

$$\begin{aligned}A \boxdot (B \boxplus C) &= A \boxdot (c + e, d + f - u) \\ &= \Big(a \cdot (c + e), u - (u - b) \cdot (u - d - f + u)\Big) \\ &= \Big(a \cdot c + a \cdot e, u - (u - b) \cdot (u - d) - (u - b) \cdot (u - f)\Big).\end{aligned}$$

On the other hand,

$$\begin{aligned}(A \boxdot B) \boxplus (A \boxdot C) &= \Big(a \cdot c, u - (u - b) \cdot (u - d)\Big) \boxplus \Big(a \cdot e, u - (u - b) \cdot (u - f)\Big) \\ &= \Big(a \cdot c + a \cdot e, u - (u - b) \cdot (u - d) - (u - b) \cdot (u - f)\Big).\end{aligned}$$

Therefore,

$$A \boxdot (B \boxplus C) = (A \boxdot B) \boxplus (A \boxdot C).$$

■

Definition 3. We say that $\boxplus_{i=1}^{n}(a_i,b_i)$ is defined if

$$\sum_{i=1}^{k} a_i \leq u, \qquad 0 \leq \sum_{i=1}^{k} b_i - (k-1)u \leq u,$$

for every $k = 1, ..., n$.

Now, we can introduce an IF-dynamical system on a product MV-algebra M.

Definition 4. By an IF-dynamical system on M, we understand a triple $(\mathcal{F}, s, \phi)$, where $s : \mathcal{F} \longrightarrow [0,1]$ is an IF-state on $\mathcal{F}$ and $\phi : \mathcal{F} \longrightarrow \mathcal{F}$ is defined by $\phi((a,b)) = (U_1(a), U_2(b))$ such that (M, m_α, U_1) and (M, m_α, U_2) are dynamical systems on M, also ϕ satisfying the following conditions:

i) If $(a_1,b_1) \boxplus (a_2,b_2)$ is defined, then $\phi((a_1,b_1)) \boxplus \phi((a_2,b_2))$ is defined and $\phi((a_1,b_1) \boxplus (a_2,b_2)) = \phi((a_1,b_1)) \boxplus \phi((a_2,b_2))$,

ii) $\phi((a_1,b_1) \boxdot (a_2,b_2)) = \phi((a_1,b_1)) \boxdot \phi((a_2,b_2))$,

iii) $\phi((u,0)) = (u,0)$,

iv) $s(\phi((a,b))) = s((a,b))$ for any $(a,b) \in \mathcal{F}$,

v) $s((a,b) \boxdot (u,0)) = s((a,b))$ for any $(a,b) \in \mathcal{F}$.

Example 2.1. Let a mapping $\phi : \mathcal{F} \longrightarrow \mathcal{F}$ be defined by $\phi((a,b)) = (U(a), b)$ such that (M, m_α, U) is a dynamical system on M. Then, $(\mathcal{F}, s_\alpha, \phi)$ is an IF-dynamical system.

3. Entropy of an IF-dynamical System

Definition 5. A finite collection $A = \{(a_1,b_1), ..., (a_n,b_n)\}$ of elements of $\mathcal{F}$ is said to be an IF-partition of an MV-algebra M iff

(1) $\boxplus_{i=1}^{n}(a_i,b_i)$ is defined,

(2) $\boxplus_{i=1}^{n}(a_i,b_i) = (u,0)$.

Lemma 2. Let $A = \{(a_1,b_1), ..., (a_n,b_n)\}$ be an IF-partition of an MV-algebra M. Then,

$$s(\boxplus_{i=1}^{n}(a_i,b_i)) = \sum_{i=1}^{n} s((a_i,b_i)).$$

Proof. Since $(a_1,b_1) \boxplus (a_2,b_2)$ is defined, it holds that $a_1 + a_2 \leqslant u$ and $0 \leqslant b_1 + b_2 - u \leqslant u$,. Thus, $(a_1,b_1) \odot (a_2,b_2) = (0,u)$. Therefore, by Definition 1,

$$s((a_1,b_1) \boxplus (a_2,b_2)) = s((a_1,b_1)) + s((a_2,b_2)).$$

Then, it follows by induction that

$$s(\boxplus_{i=1}^{n}(a_i,b_i)) = \sum_{i=1}^{n} s((a_i,b_i)).$$

■

Definition 6. Let $A = \{(a_1,b_1),...,(a_n,b_n)\}$ and $B = \{(c_1,d_1),...,(c_m,d_m)\}$ be two IF-partitions of a product MV-algebra M. Then, the common refinement of these IF-partitions is defined as the collection

$$A \vee B = \{(a_i,b_i) \boxdot (c_j,d_j) : i = 1,...,n,\ j = 1,...,m\}.$$

Definition 7. Let $A = \{(a_1,b_1),...,(a_n,b_n)\}$ and $B = \{(c_1,d_1),...,(c_m,d_m)\}$ be two IF-partitions of a product MV-algebra M. Then, B is called a refinement of A, written as $A \leqslant B$, if there exists a partition $I(1),...,I(n)$ of the set $\{1,...,m\}$ such that

$$s((a_i,b_i)) = \sum_{j \in I(i)} s((c_j,d_j)),$$

for every $i = 1,...,n$.

Proposition 1. If $A = \{(a_1,b_1),...,(a_n,b_n)\}$ and $B = \{(c_1,d_1),...,(c_m,d_m)\}$ are two IF-partitions of a product MV-algebra M, then $A \vee B$ is also an IF-partition. Furthermore, $A \leqslant A \vee B$.

Proof.

$$\begin{aligned}
&\boxplus_{i=1}^{n} \boxplus_{j=1}^{m} (a_i,b_i) \boxdot (c_j,d_j) \\
&= \boxplus_{i=1}^{n} \boxplus_{j=1}^{m} (a_i \cdot c_j, -(-b_i \cdot u) - (-d_j \cdot u) - (-b_i \cdot -d_j)) \\
&= \Bigg(\sum_{i=1}^{n} \sum_{j=1}^{m} a_i \cdot c_j, \sum_{i=1}^{n} \sum_{j=1}^{m} -(-b_i \cdot u) \\
&+ \sum_{i=1}^{n} \sum_{j=1}^{m} -(-d_j \cdot u) - \sum_{i=1}^{n} \sum_{j=1}^{m} (-b_i \cdot -d_j) - (mn-1)u \Bigg) \\
&= \Bigg((\sum_{i=1}^{n} a_i) \cdot (\sum_{j=1}^{m} c_j), -\sum_{j=1}^{m} (\sum_{i=1}^{n} (-b_i \cdot u)) \\
&- \sum_{i=1}^{n} (\sum_{j=1}^{m} (-d_j \cdot u)) - (\sum_{i=1}^{n} (-b_i)) \cdot (\sum_{j=1}^{m} (-d_j)) - (mn-1)u \Bigg) \\
&= \Big(u, m(n-1)u + n(m-1)u - (n-1)(m-1)u - (mn-1)u \Big) \\
&= (u,0).
\end{aligned}$$

Finally, let us mention that $A \vee B$ is indexed by $\{(i,j); i = 1,...,n,\ j = 1,...,m\}$. Therefore, if we put $I(i) = \{(i,1),...,(i,m)\}$, and use the equality

$$(u,0) = \boxplus_{j=1}^{m}(c_j,d_j),$$

we obtain

$$\begin{aligned} s((a_i,b_i)) &= s((a_i,b_i)\boxdot(u,0)) \\ &= s((a_i,b_i)\boxdot\left(\boxplus_{j=1}^{m}(c_j,d_j)\right)) \\ &= \sum_{j=1}^{m} s((a_i,b_i)\boxdot(c_j,d_j)) \\ &= \sum_{(i,j)\in I(i)} s((a_i,b_i)\boxdot(c_j,d_j)) \end{aligned}$$

for every $i=1,\dots,n$. It follows that $A\vee B\geqslant A$. ■

Lemma 3. If $A=\{(a_1,b_1),\dots,(a_n,b_n)\}$ is an IF-partition, then $A^\flat=\{a_1,\dots,a_n\}$, $A^\sharp=\{u-b_1,\dots,u-b_n\}$ are partitions of the MV-algebra M.

Proof. Since A is an IF-partition, it follows that

$$\begin{aligned} \boxplus_{i=1}^{n}(a_i,b_i)=(u,0) &\Longrightarrow (\sum_{i=1}^{n}a_i,\sum_{i=1}^{n}b_i-(n-1)u)=(u,0) \\ &\Longrightarrow \sum_{i=1}^{n}a_i=u, \qquad \sum_{i=1}^{n}b_i=(n-1)u \\ &\Longrightarrow \sum_{i=1}^{n}(u-b_i)=nu-(n-1)u=u. \end{aligned}$$

Therefore, $A^\flat$, $A^\sharp$ are partitions of M. ■

Lemma 4. If $A=\{(a_1,b_1),\dots,(a_n,b_n)\}$ and $B=\{(c_1,d_1),\dots,(c_m,d_m)\}$ are IF-partitions of a product MV-algebra M, then

$$(A\vee B)^\flat=A^\flat\vee B^\flat, \qquad (A\vee B)^\sharp=A^\sharp\vee B^\sharp.$$

Proof.

$$\begin{aligned} (A\vee B)^\flat &= \{a_i\cdot c_j : i=1,\dots,n,\ j=1,\dots,m\} \\ &= \{a_i : i=1,\dots,n\}\vee\{c_j : j=1,\dots,m\} \\ &= A^\flat\vee B^\flat. \end{aligned}$$

Moreover,

$$\begin{aligned} (A\vee B)^\sharp &= \{u+(-b_i\cdot u)+(-d_j\cdot u)+(-b_i\cdot -d_j) : i=1,\dots,n,\ j=1,\dots,m\} \\ &= \{(u-b_i)\cdot(u-d_j) : i=1,\dots,n,\ j=1,\dots,m\} \\ &= \{(u-b_i) : i=1,\dots,n\}\vee\{(u-d_j) : j=1,\dots,m\}=A^\sharp\vee B^\sharp. \end{aligned}$$

■

Definition 8. If A is an IF-partition, then we define the entropy of A (with respect to a given state s_α) as

$$H_\alpha(A) = (1-\alpha)H(A^\flat) + \alpha H(A^\sharp),$$

where H is the entropy of partitions in a product MV-algebra M (see equation (1)).

Proposition 2. If $A = \{(a_1,b_1),\ldots,(a_n,b_n)\}$ and $B = \{(c_1,d_1),\ldots,(c_m,d_m)\}$ are two IF-partitions, then

$$H_\alpha(A \vee B) \leq H_\alpha(A) + H_\alpha(B).$$

Proof. By lemma 4,

$$\begin{aligned} H_\alpha(A \vee B) &= (1-\alpha)H((A\vee B)^\flat) + \alpha H((A\vee B)^\sharp) \\ &= (1-\alpha)H(A^\flat \vee B^\flat) + \alpha H(A^\sharp \vee B^\sharp) \\ &\leq (1-\alpha)H(A^\flat) + \alpha H(A^\sharp) + (1-\alpha)H(B^\flat) + \alpha H(B^\sharp) \\ &= H_\alpha(A) + H_\alpha(B). \end{aligned}$$

■

Definition 9. If A and B are two IF-partitions of a product MV-algebra M, then we define the conditional entropy (with respect to a given state s_α)

$$H_\alpha(A \mid B) = (1-\alpha)H(A^\flat \mid B^\flat) + \alpha H(A^\sharp \mid B^\sharp),$$

where H is the conditional entropy of partitions in a product MV-algebra M (see equation (2)).

Proposition 3. If A, B, C are IF-partitions of a product MV-algebra M, then

i) $H_\alpha(B \vee C \mid A) = H_\alpha(B \mid A) + H_\alpha(C \mid B \vee A)$,
ii) $H_\alpha(B \vee C) = H_\alpha(B) + H_\alpha(C \mid B)$.

Proof. i) Since

$$H(B^\flat \vee C^\flat \mid A^\flat) = H(B^\flat \mid A^\flat) + H(C^\flat \mid B^\flat \vee A^\flat),$$

and

$$H(B^\sharp \vee C^\sharp \mid A^\sharp) = H(B^\sharp \mid A^\sharp) + H(C^\sharp \mid B^\sharp \vee A^\sharp),$$

we have

$$\begin{aligned} H_\alpha(B \vee C \mid A) &= (1-\alpha)H(B^\flat \vee C^\flat \mid A^\flat) + \alpha H(B^\sharp \vee C^\sharp \mid A^\sharp) \\ &= (1-\alpha)H(B^\flat \mid A^\flat) + (1-\alpha)H(C^\flat \mid B^\flat \vee A^\flat) \\ &+ \alpha H(B^\sharp \mid A^\sharp) + \alpha H(C^\sharp \mid B^\sharp \vee A^\sharp) \\ &= H_\alpha(B \mid A) + H_\alpha(C \mid B \vee A). \end{aligned}$$

ii)

$$\begin{aligned} H_\alpha(B\vee C) &= (1-\alpha)H(B^\flat \vee C^\flat) + \alpha H(B^\sharp \vee C^\sharp) \\ &= (1-\alpha)H(B^\flat) + (1-\alpha)H(C^\flat \mid B^\flat) + \alpha H(B^\sharp) + \alpha H(C^\sharp \mid B^\sharp) \\ &= H_\alpha(B) + H_\alpha(C \mid B). \end{aligned}$$

■

Proposition 4. If A is an IF-partition, then $\phi(A)$ is an IF-partition and $H_\alpha(\phi(A)) = H_\alpha(A)$.

Proof. By definition 4, since $\boxplus_{i=1}^n (a_i,b_i)$ is defined, also $\boxplus_{i=1}^n \phi((a_i,b_i))$ is defined. Then,

$$\boxplus_{i=1}^n \phi((a_i,b_i)) = \phi(\boxplus_{i=1}^n (a_i,b_i)) = \phi((u,0)) = (u,0).$$

Therefore, $\phi(A)$ is an IF-partition. It also holds that

$$\phi(A)^\flat = \{U_1(a_i) : i = 1,...,n\} = U_1(A^\flat),$$

$$\phi(A)^\sharp = \{u - U_2(b_i) : i = 1,...,n\} = U_2(A^\sharp).$$

Therefore,

$$\begin{aligned} H_\alpha(\phi(A)) &= (1-\alpha)H(\phi(A)^\flat) + \alpha H(\phi(A)^\sharp) \\ &= (1-\alpha)H(U_1(A^\flat)) + \alpha H(U_2(A^\sharp)) \\ &= (1-\alpha)H(A^\flat) + \alpha H(A^\sharp) = H_\alpha(A). \end{aligned}$$

■

Proposition 5. For any IF-partition A, there exists

$$\lim_{n\longrightarrow\infty} \frac{1}{n} H_\alpha(\bigvee_{i=0}^{n-1} \phi^i(A)).$$

Proof. By Proposition 2, $H_\alpha(B\vee C) \le H_\alpha(B) + H_\alpha(C)$ for any IF-partitions B and C. Put $a_n = H_\alpha(\bigvee_{i=0}^{n-1} \phi^i(A))$ for any $n \in \mathbb{N}$. Then, $a_{n+m} \le a_n + a_m$ for every $n,m \in \mathbb{N}$ and this property guarantees the existence of the limit

$$\lim_{n\longrightarrow\infty} \frac{1}{n} a_n = \lim_{n\longrightarrow\infty} \frac{1}{n} H_\alpha(\bigvee_{i=0}^{n-1} \phi^i(A)).$$

■

Definition 10. For every IF-partition A, we define

$$h_\alpha(\phi,A) = \lim_{n\longrightarrow\infty} \frac{1}{n} H_\alpha(\bigvee_{i=0}^{n-1} \phi^i(A)).$$

Further, we define the entropy of an IF-dynamical system by

$$h_\alpha(\phi) = \sup\{h_\alpha(\phi, A); A \text{ is an IF-partition}\}.$$

4. IF-dynamical Systems on MV-Algebras

Now we introduce an IF-dynamical system on an arbitrary MV-algebra M. We consider an algebraic structure $(\mathcal{F}, \boxplus, (u,0))$, where $\boxplus$ is a partial binary operation on $\mathcal{F}$ defined by the formula (3).

Definition 11. By an IF-dynamical system on an arbitrary MV-algebra M, we shall understand a triple $(\mathcal{F}, s, \phi)$, where $s : \mathcal{F} \longrightarrow [0,1]$ is a IF-state on $\mathcal{F}$ and $\phi : \mathcal{F} \longrightarrow \mathcal{F}$ is a mapping satisfying the following conditions:

i) If $(a_1,b_1) \boxplus (a_2,b_2)$ is defined, then $\phi((a_1,b_1)) \boxplus \phi((a_2,b_2))$ is defined and $\phi((a_1,b_1) \boxplus (a_2,b_2)) = \phi((a_1,b_1)) \boxplus \phi((a_2,b_2))$,

ii) $\phi((u,0)) = (u,0)$,

iii) $s(\phi((a,b))) = s((a,b))$ for any $(a,b) \in \mathcal{F}$.

If $A = \{a_1,...,a_n\}$ and $B = \{b_1,...,b_k\}$ are partitions of unity in M, then their common refinement is any set $S = \{c_{ij}; i = 1,...,n, j = 1,...,k\}$ of elements of M such that

$$a_i = \sum_{j=1}^{k} c_{ij}, i = 1,...,n, \qquad b_j = \sum_{i=1}^{n} c_{ij}, j = 1,...,k.$$

Lemma 5. To any partitions A, B, there exists their common refinement.

Proof. See [16]. ■

Proposition 6. To any IF-partitions A, B, there exists their common refinement.

Proof. Let $A = \{(a_1,b_1),...,(a_n,b_n)\}$ and $B = \{(t_1,d_1),...,(t_m,d_m)\}$ be two IF-partitions. By Lemma 5, $A^\flat$, $A^\sharp$, $B^\flat$ and $B^\sharp$ are partitions of MV-algebra M. By Lemma 4.2, for partitions $A^\flat$ and $B^\flat$, there exists a common refinement

$$C^\flat = \{c^\flat_{ij} : i = 1,...,n, j = 1,...,m\}$$

of elements of M such that

$$a_i = \sum_{j=1}^{m} c^\flat_{ij}, i = 1,...,n,$$

$$t_j = \sum_{i=1}^{n} c^\flat_{ij}, j = 1,...,m,$$

and for partitions $A^\sharp$ and $B^\sharp$, there exists a common refinement

$$C^\sharp = \{c^\sharp_{ij} : i = 1,...,n, j = 1,...,m\}$$

of elements of M such that

$$u - b_i = \sum_{j=1}^{m} c^\sharp_{ij}, i = 1,...,n,$$

$$u - d_j = \sum_{i=1}^{n} c^\sharp_{ij}, j = 1,...,m.$$

Now, we consider the collection

$$C = \{C_{ij} = (c^\flat_{ij}, u - c^\sharp_{ij}) : i = 1,...,n, j = 1,...,m\}.$$

We have

$$\sum_{j=1}^{l}\sum_{i=1}^{k} c^\flat_{ij} \leq \sum_{j=1}^{l} t_j \leq u,$$

and

$$\sum_{j=1}^{l}\sum_{i=1}^{k}(u - c^\sharp_{ij}) \leq \sum_{j=1}^{l}\sum_{i=1}^{k} u \leq klu,$$

for every $l = 1,...,m, k = 1,...,n$. Also, since B is an IF-partition, then $(l-1)u \leq \sum_{j=1}^{l} d_j \leq lu$, thus

$$\begin{aligned}\sum_{j=1}^{l}\sum_{i=1}^{k}(u - c^\sharp_{ij}) &= klu - \sum_{j=1}^{l}\sum_{i=1}^{k} c^\sharp_{ij}\\ &\geq klu - \sum_{j=1}^{l}(u - d_j)\\ &\geq (kl-1)u,\end{aligned}$$

for every $l = 1,...,m, k = 1,...,n$. Therefore, $\boxplus_{i=1}^{n} \boxplus_{j=1}^{m} (c^\flat_{ij}, u - c^\sharp_{ij})$ is defined as

$$\begin{aligned}\boxplus_{j=1}^{m}(c^\flat_{ij}, u - c^\sharp_{ij}) &= (\sum_{j=1}^{m} c^\flat_{ij}, \sum_{j=1}^{m}(u - c^\sharp_{ij}) - (m-1)u)\\ &= (a_i, mu - \sum_{j=1}^{m} c^\sharp_{ij} - (m-1)u) = (a_i, b_i),\end{aligned}$$

for every $i = 1,...,n$.

$$\begin{aligned}\boxplus_{i=1}^{n}(c^\flat_{ij}, u - c^\sharp_{ij}) &= (\sum_{i=1}^{n} c^\flat_{ij}, \sum_{i=1}^{n}(u - c^\sharp_{ij}) - (n-1)u)\\ &= (t_j, nu - \sum_{i=1}^{n} c^\sharp_{ij} - (n-1)u)\\ &= (t_j, d_j),\end{aligned}$$

for every $j = 1, ..., m$. Also,

$$\begin{aligned}
\boxplus_{i=1}^{n} \boxplus_{j=1}^{m} (c_{ij}^{\flat}, u - c_{ij}^{\sharp}) &= (\sum_{i=1}^{n}\sum_{j=1}^{m} c_{ij}^{\flat}, \sum_{i=1}^{n}\sum_{j=1}^{m}(u - c_{ij}^{\sharp}) - (mn-1)u) \\
&= (\sum_{i=1}^{n} a_i, mnu - \sum_{i=1}^{n}\sum_{j=1}^{m} c_{ij}^{\sharp} - (mn-1)u) \\
&= (\sum_{i=1}^{n} a_i, u - \sum_{i=1}^{n}(u - b_i)) \\
&= (u, 0).
\end{aligned}$$

Therefore, C is an IF-partition and is common refinement of A and B. ■

Definition 12. If $A = \{(a_1, b_1), ..., (a_n, b_n)\}$ and $B = \{(t_1, d_1), ..., (t_m, d_m)\}$ are two IF-partitions of an MV-algebra M, we define the entropy of A by

$$H(A) = \sum_{i=1}^{n} f(s((a_i, b_i))),$$

where $f(x) = -x \log x$, if $x > 0$, $f(0) = 0$ and the conditional entropy by

$$H_C(A|B) = \sum_{i=1}^{n}\sum_{j=1}^{m} s((t_j, d_j)) f(\frac{s(C_{ij})}{s((t_j, d_j))}).$$

We omit the $j-$terms when $s((t_j, d_j)) = 0$.

Proposition 7. Let A and B be IF-partitions of M, and C be an arbitrary common refinement of A and B. Then,

i) $H_C(A|B) \leq H(A)$,
ii) $H(C) = H(A) + H_C(B|A)$,
iii) $H(C) \leq H(A) + H(B)$.

Proof. We only prove (ii).

$$\begin{aligned}
H(C) &= \sum_{i=1}^{n}\sum_{j=1}^{m} f(s(C_{ij})) = \sum_{i=1}^{n}\sum_{j=1}^{m} f(s((a_i, b_i))\frac{s(C_{ij})}{s((a_i, b_i))}) \\
&= -\sum_{i=1}^{n}\sum_{j=1}^{m} s(C_{ij}) \log s((a_i, b_i)) - \sum_{i=1}^{n}\sum_{j=1}^{m} s(C_{ij}) \log(\frac{s(C_{ij})}{s((a_i, b_i))}) \\
&= -\sum_{i=1}^{n} \log s((a_i, b_i)) \sum_{j=1}^{m} s(C_{ij}) + \sum_{i=1}^{n}\sum_{j=1}^{m} s((a_i, b_i)) f(\frac{s(C_{ij})}{s((a_i, b_i))}) \\
&= -\sum_{i=1}^{n} s((a_i, b_i)) \log(s((a_i, b_i))) + H_C(B|A) \\
&= H(A) + H_C(B|A).
\end{aligned}$$

■

Lemma 6. For any IF-partition A, $\phi(A)$ is an IF-partition, too. Moreover, $H(\phi(A)) = H(A)$.

Proof. The proof is straightforward. ∎

Definition 13. For any IF-partition A and any positive integer n, we define

$$H_n(\phi, A) = \inf_C H(C),$$

such that C is a common refinement of $A, \phi(A), ..., \phi^{n-1}(A)$.

Proposition 8. There exists $\lim_{n\longrightarrow\infty} \frac{1}{n} H_n(\phi, A)$.

Proof. See [16]. ∎

Definition 14. The entropy of an IF-dynamical system $(\mathcal{F}, s, \phi)$ is defined by the formula

$$h(\phi) = \sup_A h(\phi, A),$$

where A is an IF-partition and

$$h(\phi, A) = \lim_{n\longrightarrow\infty} \frac{1}{n} H_n(\phi, A).$$

5. Conclusions

In this paper, we have studied dynamical systems based on IF-events in MV-algebras. We have defined a special type of the notion of entropy on this system in a product MV-algebra. Also, we have introduced the entropy of an IF-dynamical system on an arbitrary MV-algebra.

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