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Exact Solution for the Conformable Burgers' Equation by the Hopf-Cole Transform

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Abstract: In this paper, we use Hopf-Cole transform to solve conformable Burgers' equation. After applying Hopf-Cole transform to conformable Burgers' equation, we achieve conformable heat equation. Subsequently by using Fourier transform we have the exact solution of conformable Burgers' equation with fractional order.

Keywords: Hopf-Cole transform, conformable Burgers' equation, conformable derivative, conformable heat equation.

1. Introduction

The equation

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=\nu\frac{\partial^2 u}{\partial x^2}, \tag{1}$$

which is known as a one dimensional non-linear parabolic partial differential equation, was firstly seen in Bateman's article in 1918 [1]. Burgers' equation has been used as a mathematical model in various areas such as number theory, gas dynamics, heat conduction, elasticity theory [2]. Recognition of the equation in the literature has been made by means of J.M. Burgers' studies in 1939–1965. Burger used the Burgers' equation as a mathematical model especially in his studies on turbulence theory and he found the similarity of the equation to the Navier-Stokes equations since the equation contains higher-order terms which are multiplied by nonlinear small parameters [3, 4]. In 1951 Cole has shown that (1) has the typical properties of shock wave theory [5]. He has indicated that the nonlinear terms arrange the wave length and wave width, the parameter ν avoids the natural discontinuous. Also he has indicated that, if two sides of the Burgers' equation are multiplied by u and integrated in the finite domain $x_1 \leq x \leq x_2$, the equation turns into an energy equation as follows.

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$$\frac{1}{2}\int_{x_1}^{x_2}\frac{\partial}{\partial t}(u^2)dx+\frac{1}{3}u^3|_{(x_1,t)}^{(x_2,t)}=\nu u\frac{\partial u}{\partial x}|_{(x_1,t)}^{(x_2,t)}-\nu\int_{x_1}^{x_2}\left(\frac{\partial u}{\partial x}\right)^2dx.$$

E. Varoglu and L. Finn [6] stated that Burgers' equation defines a model for fluid mechanics and in this model $u(x,t)$ represents the velocity and ν represents the fluid flow based on the velocity. Blackstock [7] stated in his article that a general method has been developed which mentions finite amplitude wave problems in thermoviscous fluids and this method is based on using Burgers' equation. Also Burgers' equation has been used as a model for some physical events such as hydrodynamic waves [8] and elastic waves [9]. T. Ozis and A. Ozdes used the direct variational method to obtain solutions to Burgers' equation and show that the obtained solutions are attuned to the solutions which are obtained by other methods [10]. E. Varoğlu and L. Finn used the weighted residue method [6], J. Caldwell and P. Wanless used the finite element method [11], D.J. Evans and A.R. Abdullah used the group explicit method [12], R.C. Mittal and P. Singhal used the Galerkin method [13] to find numerical solutions of Burgers' equation.

In addition, there are various studies on the fractional Burgers' equation which is one of the non-linear equations, where the solution of the equation can be obtained. For example A. Esen *et al.* [14] used HAM to find the approximate analytical solution of Burgers' equation. In another study, E. A.-B. Abdel-Salam *et al.* [15] used the fractional Riccati expansion method to solve Burgers' equation. A. Esen and O. Taşbozan [16] used Cubic B-spline Finite Elements to obtain a numerical solution of time fractional Burgers' equation. M. Inc [17] used the variational iteration method to solve the space-time fractional Burgers' equations with initial conditions and compared the obtained solutions with the solutions that are obtained by the Adomian decomposition method.

In 1950, Hopf [18] defined the following transform to solve the Burgers' equation

$$u=-2\nu\frac{\theta_x}{\theta} \tag{2}$$

With the help of this transform Burgers' equation turns into the heat equation

$$\frac{\partial\theta}{\partial t}=\nu\frac{\partial^2\theta}{\partial t^2}, \tag{3}$$

where $\theta(x,t)$ is the solution of the heat equation and $u(x,t)$ is the solution of (1) [19]. In various studies, the solution of Burgers' equation has been obtained by the transform (2) [19]. In 1951, Cole [5] has given some theorems which express the relationship between Burgers' equation and the heat equation.

In the last decades, scientists pay great attention to fractional derivatives. By using the fractional derivative, many models which were expressed with the help of the integer-order derivative have been generalized to fractional-order derivative. These models have numerous applications in various fields, such as physics, biology, engineering, signal processing, and control theory, finance and fractal dynamics [20, 21, 22]. As a result of this attention and huge application area, the scientists expressed some definitions of fractional derivatives such as Caputo fractional derivative, Riemann-Liouville fractional derivative, Grünwald–Letnikov fractional derivative etc. The most popular ones are;

1. Riemann-Liouville Definition: If n is a positive integer and $\alpha \in [n-1,n)$, α derivative of function f is given by

$$D_a^{\alpha}(f)(t) = \frac{1}{\Gamma(n-a)} \frac{d^n}{dt^n} \int_a^t \frac{f(x)}{(t-x)^{\alpha-n+1}} dx.$$

2. Caputo Defintion: If n is a positive integer and $\alpha \in [n-1,n)$, α derivative of function f is given by

$$D_a^{\alpha}(f)(t) = \frac{1}{\Gamma(n-a)} \int_a^t \frac{f^{(n)}(x)}{(t-x)^{\alpha-n+1}} dx.$$

Nowadays, a new definition called *Conformable Fractional Derivative* is stated by R. Khalil *et al.* [23].

Definiton Let $f : [0,\infty) \to \mathbb{R}$ be a function. α^{th} order *"conformable fractional derivative"* of f is defined by

$$T_{\alpha}(f)(t) = \lim_{\varepsilon \to 0} \frac{f(t+\varepsilon t^{1-\alpha}) - f(t)}{\varepsilon}$$

for all $t > 0, \alpha \in (0,1)$. If f is α-differentiable in some $(0,a), a > 0$ and $\lim_{t \to 0^+} f^{(\alpha)}(t)$ exists then define $f^{(\alpha)}(0) = \lim_{t \to 0^+} f^{(\alpha)}(t)$.

Some properties of this new definition are given in the following theorem [23].

Theorem 1. Let $\alpha \in (0,1]$ and f,g be α-differentiable at point $t > 0$. Then,

1. $T_{\alpha}(cf + dg) = cT_{\alpha}(f) + cT_{\alpha}(g)$ for all $a,b \in \mathbb{R}$,
2. $T_{\alpha}(t^p) = pt^{p-\alpha}$ for all $p \in \mathbb{R}$,
3. $T_{\alpha}(\lambda) = 0$ for all constant functions $f(t) = \lambda$,
4. $T_{\alpha}(fg) = fT_{\alpha}(g) + gT_{\alpha}(f)$,
5. $T_{\alpha}\left(\frac{f}{g}\right) = \frac{gT_{\alpha}(f) - fT_{\alpha}(g)}{g^2}$,
6. If f is differentiable, then $T_{\alpha}(f)(t) = t^{1-\alpha}\frac{df}{dt}$.

Recently, there are numerous studies on this new definition. For instance, T. Abdeljawad [24] gave conformable versions of the chain rule, integration by parts, Taylor power series expansions and Laplace, transform. In an other study, T. Abdeljawad *et al.* [25] introduce a conformable fractional semigroup of operators whose generator will be the fractional derivative of the semigroup at $t = 0$. Lastly D.R. Anderson, and D.J. Ulness [26] introduced a more precise definition of a conformable derivative motivated by a proportional-derivative (PD) controller.

2. The Conformable Burgers' Equation

Let us consider the equation

$$\frac{\partial^\alpha u}{\partial t^\alpha} + u\frac{\partial u}{\partial x} - v\frac{\partial^2 u}{\partial x^2} = 0,\ a < x < b,\ t > 0, \tag{4}$$

which is known as Burgers' equation, where $\frac{\partial^\alpha u}{\partial t^\alpha}$ means conformable derivative of function $u(x,t)$. It is known that (4) is the one-dimensional non-linear time fractional partial differential equation. By using the Hopf-Cole transformation,

$$u(x,t) = -2v\left(Q_x/Q\right), \tag{5}$$

the Burgers' equation turns into the time fractional heat equation

$$\frac{\partial^\alpha Q}{\partial t^\alpha} = v\frac{\partial^2 Q}{\partial x^2} \tag{6}$$

Then, applying the Fourier transform, (6) can be written in the following form

$$\frac{\partial^\alpha Q(w,t)}{\partial t^\alpha} + vw^2 Q = 0, \tag{7}$$

where $\frac{\partial^\alpha Q(w,t)}{\partial t^\alpha}$ denotes the $\alpha-$order conformable derivative. With the help of 6. in Theorem 1, we can write (7) as follows.

$$t^{1-\alpha}\frac{\partial Q}{\partial t} + vw^2 Q = 0. \tag{8}$$

One can easily see that the solution of the first order differential equation is

$$Q(w,t) = e^{-\frac{w^2 v}{\alpha}t^\alpha}. \tag{9}$$

Applying the inverse Fourier transform to (8), we get

$$Q(x,t) = \frac{e^{-\frac{t^{-\alpha}x^2\alpha}{4v}}}{\sqrt{\frac{2t^\alpha v}{\alpha}}}.$$

Again using the Hopf-Cole transformation (5) the solution of (4) can be obtained as

$$u(x,t) = \alpha x t^{-\alpha}.$$

3. Conclusion

In this paper we deliberate over the new exact solution of the conformable Burgers' equation, which is known as a one dimensional non-linear time fractional partial differential equation. It is known that Burgers' equation is one of the rare non-linear equations whose solution can be obtained analytically. Therefore, Burgers' equation has a great importance in applied sciences.

The Hopf-Cole transform and the conformable derivative definition which is a new local (conformable) fractional derivative definition are used for the exact solution of the time fractional Burgers' equation. We can say that this local (conformable) fractional derivative definition is a convenient definition in the exact solution procedure of fractional differential equations.

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