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RECURRENCE RELATION FOR THE MOMENTS OF ORDER STATISTICS FROM A BETA-PARETO DISTRIBUTION

Hossein Jabbari Khamnei*

Department of Statistics, Faculty of Mathematical Sciences,
University of Tabriz, Tabriz, Iran.

Roghaye Makouyi

Payame Noor University of Tabriz, Tabriz, Iran.
Tabriz University of Medical Sciences, Tabriz, Iran.

Abstract: In this paper, a novel cumulative distribution function ($c.d.f.$) for beta-pareto (BP) distribution, through two distinct practical frames, is developed. However, the presented models are obviously more pragmatic than the ones being demonstrated in previous works, in the case of extending the further relations. Then, using the exhibited c.d.f.s, certain recurrence relations for the single and product moments of the order statistics of a random sample of size n arising from beta-Pareto distribution are derived.

Key words: Order statistics; Single and product moments; Recurrence relations; Beta-Pareto.
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1. Introduction

The class of generalized beta (GB) distribution was first introduced by [6] through its cumulative distribution function ($c.d.f.$). The $c.d.f.$ of the class of GB distribution is defined by

$$F(x) = \frac{1}{B(\alpha,\beta)} \int_0^{G(x)} t^{\alpha-1}(1-t)^{\beta-1} dt, \quad \alpha > 0, \beta > 0, \tag{1.1}$$

The probability density function ($p.d.f.$), when X is continuous, for the GB distribution is given by

$$f(x) = \frac{1}{B(\alpha,\beta)} g(x) G(x)^{\alpha-1} (1 - G(x))^{\beta-1}, \tag{1.2}$$

where

$$B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}.$$

Since the paper by [6] in which the normal distribution was defined and studied, many papers have appeared in this class. These include beta-frechet [11], beta-weibull [7], beta-pareto (BP)[1], β-birnbaum-saunders [4], and beta-cauchy [2]. The beta-Pareto (BP) distribution [1] was applied to fit flood data.

The order statistics have been extensively studied in the previous literature. For an excellent review, we refer to [2], [3] and [5]. The use of recurrence relations for the moments of order statistics is quite well-known in statistical literature (see, for example, [2];[10]). For improved forms of these

* Corresponding author. E-mail address: h_jabbari@tabrizu.ac.ir

results, it can be seen that [12] and [13]. [3] have reviewed many recurrence relations and identities for the moments of order statistics arising from several specific continuous distributions such as normal, Cauchy, logistic, gamma and exponential. Hence, the aim of this paper is to consider order statistics of a random sample of size n drawn from BP distribution and derive some recurrence relations for the single and product moments of these order statistics.

Let $g(x)$ and $G(x)$ respectively be the *p.d.f.* and *c.d.f.* of pareto distribution, are given by

$$g(x) = \frac{k\theta^k}{x^{k+1}}, \qquad k > 0, \theta > 0, x \geq \theta, \tag{1.3}$$

$$G(x) = 1 - \left(\frac{x}{\theta}\right)^{-k}, \qquad k > 0,\, \theta > 0,\, x \geq \theta. \tag{1.4}$$

By replacing (1.3) and (1.4) in (1.2), the p.d.f. for the BP random variable is given by

$$f(x) = \frac{k}{\theta B(\alpha,\beta)} \left\{1 - \left(\frac{x}{\theta}\right)^{-k}\right\}^{\alpha-1} \left(\frac{x}{\theta}\right)^{-k\beta-1}, \qquad \alpha, \beta, \theta, k > 0, x \geq \theta. \tag{1.5}$$

This case is denoted by $X \sim BP(\alpha, \beta, \theta, k)$. The *c.d.f.* of BP random variable is denoted as $F(x)$. Let $X_{1:n} \leq X_{2:n} \leq ... \leq X_{n:n}$ be the order statistics of a random sample ($r.s.$) of size n drawn from the BP distribution with *p.d.f.* (1.5). Then the p.d.f. for the $i-th$ order statistic, $X_{i:n}$, is given by

$$f_{i:n}(x) = i\binom{n}{i} F(x)^{i-1}[1 - F(x)]^{n-i} f(x), \tag{1.6}$$

where

$$\binom{n}{i} = \frac{n!}{i!(n-i)!}.$$

The $r-th$ moment of $X_{i:n}$ for $1 \leq i \leq n$ is as follows,

$$\mu_{i:n}^{(r)} = i\binom{n}{i} \int_{\theta}^{\infty} x^r F(x)^{i-1}[1 - F(x)]^{n-i} f(x) dx. \tag{1.7}$$

The joint *p.d.f.* for the $i-th$ and $j-th$ order statistic, $(X_{i:n}, X_{j:n})$ is given by

$$f_{i,j:n}(x,y) = c_{i,j:n} F(x)^{i-1}[F(y) - F(x)]^{j-i-1}[1 - F(y)]^{n-j} f(x) f(y). \tag{1.8}$$

The $(r,s)-th$ product moment of $(X_{i:n}, X_{j:n})$ for $n \geq 2$ and $1 \leq i < j \leq n$, denoted by $\mu_{i,j:n}$, is given by

$$\mu_{i,j:n}^{(r,s)} = c_{i,j:n} \int_{\theta}^{\infty} \int_{x}^{\infty} x^r y^s F(x)^{i-1}[F(y) - F(x)]^{j-i-1}[1 - F(y)]^{n-j} f(x) f(y) dy dx. \tag{1.9}$$

where

$$c_{i,j:n} = \frac{n!}{(i-1)!(j-i-1)!(n-j)!}.$$

In the following sections, we consider order statistics of a $r.s.$ of size n arising from BP distribution with integer values for the shape parameters α and β. In section 2, we provide some extensions and properties of the *c.d.f.* for the BP distribution through two distinct frames. In section 3, some recurrence relations on the single moments of these order statistics are derived. Section 4 contains some discussions for the recurrence relations on the product moments of the order statistics.

REMARK 1. In this article for any real number α and any natural k, we use the notation $\binom{\alpha}{k}$ for the generalized binomial coefficient

$$\frac{\alpha(\alpha-1)...(\alpha-k-1)}{k!}.$$

Note that by Newton's generalization of binomial expansion, for any x with $|x|<1$, we have

$$(1+x)^{\alpha}=\sum_{k=0}^{\infty}\binom{\alpha}{k}x^{k}.$$

2. Some extensions and properties

In this section we present some representations of the $c.d.f.$ of BP distribution. The mathematical relation given in below will be useful in this, and next, section. If α is a positive real non-integer and $|z|<1$, then ([8], p. 25)

$$(1-z)^{\alpha-1}=\sum_{j=0}^{\infty}w_j z^j,$$

and if α is a positive real integer, then the upper of this summation stops at $\alpha-1$, where

$$w_j=(-1)^j\binom{\alpha-1}{j}=\frac{(-1)^j\Gamma(\alpha)}{\Gamma(\alpha-j)\Gamma(j+1)}.$$

PROPOSITION 1. We can express (1.1) as a mixture of distribution function of BP distribution as follows: If $\alpha,\beta,\theta,k>0$ and j is positive integer we have,

$$F(x)=\sum_{j=0}^{\infty}C_1x^{-k(j+\beta)}+C_2, \tag{2.1}$$

where

$$C_1=\frac{\theta^{k(j+\beta)}(-1)^{j+1}\binom{\alpha-1}{j}}{B(\alpha,\beta)(j+\beta)} \quad \& \quad C_2=\sum_{j=0}^{\infty}\frac{(-1)^j\binom{\alpha-1}{j}}{B(\alpha,\beta)(j+\beta)}. \tag{2.2}$$

PROOF. The following relations are presented using (1.5) and the generalized binomial expansion by considering α is a positive real non-integer and $\left|\left(\frac{t}{\theta}\right)^{-k}\right|<1$ ([8], p. 25).

$$\begin{aligned}
F(x) &= \int_{\theta}^{x}\tfrac{k}{\theta B(\alpha,\beta)}\left\{1-\left(\tfrac{t}{\theta}\right)^{-k}\right\}^{\alpha-1}\left(\tfrac{t}{\theta}\right)^{-k\beta-1}dt\\
&= \tfrac{k}{B(\alpha,\beta)}\textstyle\sum_{j=0}^{\infty}\binom{\alpha-1}{j}(-1)^j\theta^{k(j+\beta)}\int_{\theta}^{x}t^{-k(j+\beta)-1}dt\\
&= \tfrac{k}{B(\alpha,\beta)}\textstyle\sum_{j=0}^{\infty}\binom{\alpha-1}{j}(-1)^j\theta^{k(j+\beta)}\left\{\tfrac{(-1)}{k(j+\beta)}\left(x^{-k(j+\beta)}-\theta^{-k(j+\beta)}\right)\right\}\\
&= \textstyle\sum_{j=0}^{\infty}\tfrac{\theta^{k(j+\beta)}(-1)^{j+1}\binom{\alpha-1}{j}}{B(\alpha,\beta)(j+\beta)}x^{-k(j+\beta)}+\sum_{j=0}^{\infty}\tfrac{(-1)^j\binom{\alpha-1}{j}}{B(\alpha,\beta)(j+\beta)}.
\end{aligned}$$

The simplification of the preceding equation results in driven of (2.1).

PROPOSITION 2. We can express (1.1) as a mixture of distribution function of BP distribution as follows:

$$(I) \quad F(x) = \sum_{t=0}^{\infty} b_t G(x)^t,$$

$$(II) \quad F(x)^n = \sum_{t=0}^{\infty} d_{n,t} G(x)^t,$$

where

$$b_t = \sum_{j=0}^{\infty} \sum_{l=t}^{\infty} p_j (-1)^{l+t} \binom{\alpha+j}{l} \binom{l}{t} \quad \& \quad p_j = \frac{(-1)^j \Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta-j)\Gamma(j+1)(\alpha+j)},$$

and $d_{n,t}$; $t = 1, 2, ...$ are the coefficients and are easily determined from the recurrence equation

$$d_{n,t} = (tb_0)^{-1} \sum_{m=1}^{t} [m(n+1) - t] b_m d_{n,t-m},$$

and $d_{n,0} = b_0^n$. Hence, $d_{n,t}$ comes directly from $d_{n,0}, ..., d_{n,t-1}$.

PROOF. (I) Considering [6] and by using [8]

$$\begin{aligned} F(x) &= \tfrac{1}{B(\alpha,\beta)} \int_0^{G(x)} t^{\alpha-1}(1-t)^{\beta-1} dt \\ &= \tfrac{1}{B(\alpha,\beta)} \sum_{j=0}^{\infty} \tbinom{\beta-1}{j} (-1)^j \int_0^{G(x)} t^{\alpha+j-1} dt \\ &= \sum_{j=0}^{\infty} \tfrac{(-1)^j}{B(\alpha,\beta)(\alpha+j)} \tbinom{\beta-1}{j} G(x)^{\alpha+j} \\ &= \sum_{j=0}^{\infty} p_j G(x)^{\alpha+j}, \end{aligned}$$

where

$$p_j = \frac{(-1)^j \binom{\beta-1}{j}}{B(\alpha,\beta)(\alpha+j)} = \frac{(-1)^j \Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta-j)\Gamma(j+1)(\alpha+j)},$$

also

$$\begin{aligned} G(x)^{\alpha+j} &= [1 - \{1 - G(x)\}]^{\alpha+j} \\ &= \sum_{l=0}^{\infty} (-1)^l \tbinom{\alpha+j}{l} (1 - G(x))^l \\ &= \sum_{l=0}^{\infty} (-1)^l \tbinom{\alpha+j}{l} \sum_{t=0}^{l} (-1)^t \tbinom{l}{t} G(t)^t, \end{aligned}$$

replacing $\sum_{l=0}^{\infty} \sum_{t=0}^{l}$ by $\sum_{t=0}^{\infty} \sum_{l=t}^{\infty}$ we obtain

$$G(x)^{\alpha+j} = \sum_{t=0}^{\infty} \sum_{l=t}^{\infty} (-1)^{l+t} \tbinom{\alpha+j}{l} \tbinom{l}{t} G(x)^t,$$

therefore

$$F(x) = \sum_{j=0}^{\infty} \sum_{t=0}^{\infty} \sum_{l=t}^{\infty} p_j (-1)^{l+t} \binom{\alpha+j}{l} \binom{l}{t} G(x)^t = \sum_{t=0}^{\infty} b_t G(x)^t,$$

and we obtain the result (I).

(II) Here and henceforth, we use an equation by [8], for a power series raised to a positive integer n

$$\left(\sum_{t=0}^{\infty} b_t x^t\right)^n = \sum_{t=0}^{\infty} d_{n,t} x^t,$$

with this description, the proposition (II) is confirmed.

3. Recurrence relations for single moments

Throughout this section we assume that both α and β are positive non-integers. Now we prove the following results which express $\mu_{i:n}^{(r)}$, $1 \leq i \leq n$, in terms of the single moments of BP order statistics.

THEOREM 1. *Let $n \geq 2$ and $r \geq 1$ be integers. If $\alpha, \beta, \theta, k > 0$, we can write*

$$\mu_{1:n}^{(r)} = \frac{n}{n-1}\left\{(1-C_2)\mu_{1:n-1}^{(r)} - \sum_{j=0}^{\infty} C_1 \mu_{1:n-1}^{(r-k(j+\beta))}\right\}, \tag{3.1}$$

where C_1 and C_2 according to the (2.2).

PROOF. For $n \geq 2$, we may write the following using (1.6) and (1.7),

$$\begin{aligned}
\mu_{1:n}^{(r)} &= E\left(X_{1:n}^r\right) = n\int_\theta^\infty x^r \left(1-F(x)\right)^{n-1} f(x)dx \\
&= n\left\{\int_\theta^\infty x^r\left(1-F(x)\right)^{n-2} f(x)dx - \int_\theta^\infty x^r \left(1-F(x)\right)^{n-2} F(x) f(x)dx\right\} \\
&= n\left\{\left(\tfrac{1}{n-1}\right)\mu_{1:n-1}^{(r)} - \int_\theta^\infty x^r\left(1-F(x)\right)^{n-2}\left(\textstyle\sum_{j=0}^\infty C_1 x^{-k(j+\beta)} + C_2\right) f(x)dx\right\} \\
&= \tfrac{n}{n-1}\left\{(1-C_2)\mu_{1:n-1}^{(r)} - \textstyle\sum_{j=0}^\infty C_1 \mu_{1:n-1}^{(r-k(j+\beta))}\right\}.
\end{aligned}$$

THEOREM 2. *Let n and i be integers such that $n \geq 2$ and $2 \leq i \leq n$. If $\alpha, \beta, \theta, k > 0$, then we have*

$$\mu_{i:n}^{(r)} = \frac{n}{i-1}\left\{\sum_{j=0}^{\infty} C_1 \mu_{i-1:n-1}^{(r-k(j+\beta))} + C_2 \mu_{i-1:n-1}^{(r)}\right\}, \tag{3.2}$$

where C_1 and C_2 according to the (2.2).

PROOF. By using (1.7) and (2.1) we have,

$$\begin{aligned}
\mu_{i:n}^{(r)} &= E\left(X_{i:n}^{r}\right) = i\tbinom{n}{i}\int_{\theta}^{\infty} x^{r} F(x)^{i-1}\left(1-F(x)\right)^{n-i} f(x)dx \\
&= i\tbinom{n}{i}\int_{\theta}^{\infty} x^{r} F(x)F(x)^{i-2}\left(1-F(x)\right)^{n-i} f(x)dx \\
&= i\tbinom{n}{i}\int_{\theta}^{\infty} x^{r}\left(\textstyle\sum_{j=0}^{\infty} C_1 x^{-k(j+\beta)} + C_2\right) F(x)^{i-2}\left(1-F(x)\right)^{n-i} f(x)dx \\
&= \tfrac{n}{i-1}\left\{\textstyle\sum_{j=0}^{\infty} C_1 \mu_{i-1:n-1}^{(r-k(j+\beta))} + C_2 \mu_{i-1:n-1}^{(r)}\right\}.
\end{aligned}$$

Remark 2. For $n \geq 2$,

$$\mu_{n:n}^{(r)} = \frac{n}{n-1}\left\{\sum_{j=0}^{\infty} C_1 \mu_{n-1:n-1}^{(r-k(j+\beta))} + C_2 \mu_{n-1:n-1}^{(r)}\right\}. \tag{3.3}$$

The value of $\mu_{i:n}^{(r)}$ is presented in Table 1, for n up to 10, r up to 4, $\alpha = 0.5$, $\beta = 2$, $k = 2$ and $\theta = 3$.

TABLE 1. The moments of BP order statistics for n up to 10, $\alpha = 0.5$, $\beta = 2$, $k = 2$ and $\theta = 3$.

n	i	$r=1$	$r=2$	$r=3$	$r=4$
1	1	3.5343	13.4990	63.1312	621.5891
2	1	3.1686	10.1250	32.7345	107.6848
	2	3.9000	16.8731	93.5280	1135.4935
3	1	3.0860	9.5464	29.6162	92.2043
	2	3.3337	11.2821	38.9712	138.6457
	3	4.1832	19.6685	120.8064	1633.9174
4	1	3.0529	9.3295	28.5409	87.4219
	2	3.1853	10.1973	32.8419	106.5516
	3	3.4820	12.3670	45.1005	170.7398
	4	4.4169	22.1024	146.0417	2121.6432
5	1	3.0361	9.2222	28.0273	85.2257
	2	3.1203	9.7583	30.5955	96.2064
	3	3.2828	10.8558	36.2115	122.0693
	4	3.6148	13.3744	51.0264	203.1868
	5	4.6174	24.2844	169.7955	2601.2573
6	1	3.0263	9.1608	27.7380	84.0132
	2	3.0852	9.5296	29.4737	91.2885
	3	3.1906	10.2157	32.8390	106.0421
	4	3.3750	11.4959	39.5841	138.0965
	5	3.7347	14.3137	56.7476	235.7320
	6	4.7939	26.2785	192.4051	3074.3623
7	1	3.0200	9.1220	27.5576	83.2660
	2	3.0638	9.3933	28.8202	88.4962
	3	3.1387	9.8706	31.1076	98.2693
	4	3.2598	10.6759	35.1474	116.4058
	5	3.4615	12.1109	42.9117	154.3646
	6	3.8440	15.1948	62.2819	268.2790
	7	4.9523	28.1258	214.0923	3542.0429
8	1	3.0158	9.0959	27.4370	82.7703
	2	3.0497	9.3046	28.4016	86.7357
	3	3.1061	9.6591	30.0759	93.7779
	4	3.1931	10.2230	32.8272	105.7551
	5	3.3266	11.1288	37.4676	127.0565
	6	3.5424	12.7001	46.1782	170.7494
	7	3.9445	16.0263	67.6498	300.7888
	8	5.0962	29.8543	235.0126	4005.0792
9	1	3.0128	9.0775	27.3522	82.4235
	2	3.0399	9.2435	28.1156	85.5452
	3	3.0840	9.5187	29.4026	90.9025
	4	3.1501	9.9400	31.4225	99.5287
	5	3.2467	10.5767	34.5830	113.5381
	6	3.3905	11.5704	39.7752	137.8712
	7	3.6184	13.2650	49.3797	187.1885
	8	4.0377	16.8153	72.8698	333.2461
	9	5.2285	31.4841	255.2804	4464.0583
10	1	3.0106	9.0639	27.2902	82.1707
	2	3.0327	9.1994	27.9107	84.6984
	3	3.0684	9.4200	28.9353	88.9321
	4	3.1206	9.7490	30.4932	95.5002
	5	3.1944	10.2267	32.8163	105.5715
	6	3.2990	10.9268	36.3497	121.5048
	7	3.4514	11.9994	42.0589	148.7822
	8	3.6899	13.8075	52.5172	203.6483
	9	4.1246	17.5672	77.9580	365.6455
	10	5.3512	33.0305	274.9829	4919.4375

4. Recurrence relations for the product moments

In this section we prove the following results which express $\mu_{r,s:n}$, $1 \le r < s \le n$, in terms of the product moments of BP order statistics.

THEOREM 3. *Let $\alpha, \beta, \theta, k > 0$. Then for $n \ge 2$ we have,*

$$\mu_{n-1,n:n} = n\left\{\left(\mu_{1:1}^{(1)} - D_2\right)\mu_{n-1:n-1}^{(1)} - \sum_{j=0}^{\infty} D_1 \mu_{n-1:n-1}^{(2-k(j+\beta))}\right\}, \tag{4.1}$$

where

$$D_1 = \frac{\theta^{k(\beta+j)}(-1)^{j+1}\binom{\alpha-1}{j}}{B(\alpha,\beta)(\beta+j-\frac{1}{k})} \quad \& \quad D_2 = \sum_{j=0}^{\infty}\frac{\theta(-1)^j\binom{\alpha-1}{j}}{B(\alpha,\beta)(\beta+j-\frac{1}{k})}. \tag{4.2}$$

PROOF. From (1.8) we may write for $n \ge 2$,

$$\begin{aligned}\mu_{n-1,n:n} = E(X_{n-1:n}X_{n:n}) &= n(n-1)\int_\theta^\infty \int_x^\infty xyF(x)^{n-2}f(x)f(y)dydx \\ &= n(n-1)\int_\theta^\infty xF(x)^{n-2}f(x)\left\{\int_x^\infty yf(y)dy\right\}dx \\ &= n(n-1)\int_\theta^\infty xF(x)^{n-2}f(x)I_x dx, \end{aligned} \tag{4.3}$$

$$I_x = \int_x^\infty yf(y)dy = \int_\theta^\infty yf(y)dy - \int_\theta^x yf(y)dy = \mu_{1:1}^{(1)} - P_x, \tag{4.4}$$

by using (1.5) and (4.4), we get

$$\begin{aligned}P_x = \int_\theta^x yf(y)dy &= \int_\theta^x y\frac{k}{\theta B(\alpha,\beta)}\left\{1-\left(\frac{y}{\theta}\right)^{-k}\right\}^{\alpha-1}\left(\frac{y}{\theta}\right)^{-k\beta-1}dy \\ &= \sum_{j=0}^{\infty}\binom{\alpha-1}{j}(-1)^j\frac{k}{\theta B(\alpha,\beta)}\theta^{k(j+\beta)+1}\int_\theta^x y^{-k(j+\beta)}dy \\ &= \sum_{j=0}^{\infty}\binom{\alpha-1}{j}(-1)^j\frac{k}{B(\alpha,\beta)}\theta^{k(j+\beta)}\frac{(-1)}{k(j+\beta)-1}\left\{x^{-k(j+\beta)+1} - \theta^{-k(j+\beta)+1}\right\} \\ &= \sum_{j=0}^{\infty}\frac{\binom{\alpha-1}{j}(-1)^{j+1}\theta^{k(j+\beta)}}{B(\alpha,\beta)(j+\beta-\frac{1}{k})}x^{-k(j+\beta)+1} + \sum_{j=0}^{\infty}\frac{\binom{\alpha-1}{j}(-1)^j\theta}{B(\alpha,\beta)(j+\beta-\frac{1}{k})} \\ &= \sum_{j=0}^{\infty} D_1 x^{-k(j+\beta)+1} + D_2, \end{aligned} \tag{4.5}$$

where D_1 and D_2 according to the (4.2). If we insert P_x in the (4.4) then we can write

$$I_x = \mu_{1:1}^{(1)} - \sum_{j=0}^{\infty} D_1 x^{-k(j+\beta)+1} + D_2,$$

by using I_x and (4.3), the theorem 3 proved.

THEOREM 4. *Theorem 4: Let $\alpha, \beta, \theta, k > 0$. Then for $n \ge 2$ and $1 \le i < j \le n$ we have,*

$$\mu_{i,j:n}^{(r,s)} = \sum_{m=0}^{j-i-1}(-1)^m\binom{j-i-1}{m}c_{i,j:n}\frac{1}{c}\left\{\left(\frac{(n-j)!}{(n-c)!}\mu_{j-c:n-c}^{(s)} - A_1\right)\mu_{c:c}^{(r)} - \sum_{w=0}^{\infty}A_2\mu_{c:c}^{(-k(w+\beta)+s+r)}\right\}, \tag{4.6}$$

where $c = i + m$ *and*

$$A_1 = \sum_{w=0}^{\infty}\sum_{l=0}^{\infty}\sum_{t=0}^{\infty} d_{j-c+l-1,t}\frac{1}{B(\alpha,\beta)}(-1)^{l+w}\binom{t+\alpha-1}{w}\binom{n-j}{l}\frac{\theta^s}{w+\beta-\frac{1}{k}}, \quad (4.7)$$

$$A_2 = \sum_{l=0}^{\infty}\sum_{t=0}^{\infty} d_{j-c+l-1,t}\frac{1}{B(\alpha,\beta)}(-1)^{l+w+1}\binom{t+\alpha-1}{w}\binom{n-j}{l}\frac{\theta^{k(w+\beta)+s}}{w+\beta-\frac{1}{k}}. \quad (4.8)$$

PROOF. We know that

$$[F(y)-F(x)]^{j-i-1} = \sum_{m=0}^{j-i-1}(-1)^m\binom{j-i-1}{m}F(x)^m F(y)^{j-i-1-m},$$

then we have

$$\begin{aligned}
\mu_{i,j:n}^{(r,s)} &= \sum_{m=0}^{j-i-1}(-1)^m\binom{j-i-1}{m}c_{i,j:n}\int_\theta^\infty\int_x^\infty x^r y^s F(x)^{i+m-1}F(y)^{j-i-1-m}[1-F(y)]^{n-j}f(x)f(y)dydx \\
&= \sum_{m=0}^{j-i-1}(-1)^m\binom{j-i-1}{m}c_{i,j:n}\int_\theta^\infty x^r F(x)^{c-1}f(x)I_x dx, \quad (4.9)
\end{aligned}$$

where

$$\begin{aligned}
I_x &= \int_x^\infty y^s F(y)^{j-c-1}[1-F(y)]^{n-j}f(y)dy \\
&= \int_\theta^\infty y^s F(y)^{j-c-1}[1-F(y)]^{n-j}f(y)dy - \int_\theta^x y^s F(y)^{j-c-1}[1-F(y)]^{n-j}f(y)dy \\
&= \frac{1}{(j-c)\binom{n-c}{j-c}}\mu_{j-c:n-c}^{(s)} - \int_\theta^x y^s F(y)^{j-c-1}[1-F(y)]^{n-j}f(y)dy \\
&= \frac{1}{(j-c)\binom{n-c}{j-c}}\mu_{j-c:n-c}^{(s)} - P_x, \quad (4.10)
\end{aligned}$$

from (1.5) and (4.10) we can write,

$$\begin{aligned}
P_x &= \int_\theta^x y^s F(y)^{j-c-1}[1-F(y)]^{n-j}f(y)dy \\
&= \int_\theta^x y^s F(y)^{j-c-1}\sum_{l=0}^{\infty}(-1)^l\binom{n-j}{l}F(y)^l f(y)dy \\
&= \sum_{l=0}^{\infty}(-1)^l\binom{n-j}{l}\int_\theta^x y^s F(y)^{j-c+l-1}f(y)dy,
\end{aligned}$$

by using (I) and (II) from proposition (II) we have

$$F(y)^{j-c+l-1} = \left(\sum_{t=0}^{\infty}b_t G(y)^t\right)^{j-c+l-1} = \sum_{t=0}^{\infty}d_{j-c+l-1,t}G(y)^t = \sum_{t=0}^{\infty}d_{j-c+l-1,t}\left(1-\left(\frac{y}{\theta}\right)^{-k}\right)^t,$$

so

$$\begin{aligned}
P_x &= \sum_{l=0}^{\infty}(-1)^l \binom{n-j}{l} \int_{\theta}^{x} y^s F(y)^{j-c+l-1} f(y)dy \\
&= \sum_{l=0}^{\infty}(-1)^l \binom{n-j}{l} \int_{\theta}^{x} y^s \sum_{t=0}^{\infty} d_{j-c+l-1,t}\left(1-\left(\frac{y}{\theta}\right)^{-k}\right)^t f(y)dy \\
&= \sum_{l=0}^{\infty}\sum_{t=0}^{\infty} d_{j-c+l-1,t}(-1)^l \binom{n-j}{l} \int_{\theta}^{x} y^s \left(1-\left(\frac{y}{\theta}\right)^{-k}\right)^t \frac{k}{\theta B(\alpha,\beta)}\left\{1-\left(\frac{y}{\theta}\right)^{-k}\right\}^{\alpha-1}\left(\frac{y}{\theta}\right)^{-k\beta-1} dy \\
&= \sum_{w=0}^{\infty}\sum_{l=0}^{\infty}\sum_{t=0}^{\infty} d_{j-c+l-1,t}(-1)^{l+w} \frac{k}{\theta B(\alpha,\beta)}\binom{n-j}{l}\binom{t+\alpha-1}{w}\theta^{k(w+\beta)+1}\int_{\theta}^{x} y^{-k(w+\beta)+s-1}dy \\
&= \sum_{w=0}^{\infty}\sum_{l=0}^{\infty}\sum_{t=0}^{\infty} d_{j-c+l-1,t}(-1)^{l+w} \frac{k}{\theta B(\alpha,\beta)}\binom{n-j}{l}\binom{t+\alpha-1}{w}\theta^{k(w+\beta)+1}(-1) \\
&\qquad \left(\frac{1}{k(w+\beta)-s}\right)\left(x^{-k(w+\beta)+s}-\theta^{-k(w+\beta)+s}\right) \\
&= \sum_{w=0}^{\infty}\sum_{l=0}^{\infty}\sum_{t=0}^{\infty} d_{j-c+l-1,t}(-1)^{l+w} \frac{1}{\theta B(\alpha,\beta)}\binom{n-j}{l}\binom{t+\alpha-1}{w}\left(\frac{\theta^s}{w+\beta-\frac{1}{k}}\right) \\
&\qquad + \sum_{w=0}^{\infty}\sum_{l=0}^{\infty}\sum_{t=0}^{\infty} d_{j-c+l-1,t}(-1)^{l+w} \frac{1}{\theta B(\alpha,\beta)}\binom{n-j}{l}\binom{t+\alpha-1}{w}\left(\frac{\theta^{k(w+\beta)+s}}{w+\beta-\frac{1}{k}}\right)x^{-k(w+\beta)+s} \\
&= A_1 + \sum_{w=0}^{\infty} A_2 x^{-k(w+\beta)+s},
\end{aligned}$$

where

$$A_1 = \sum_{w=0}^{\infty}\sum_{l=0}^{\infty}\sum_{t=0}^{\infty} d_{j-c+l-1,t}(-1)^{l+w} \frac{1}{B(\alpha,\beta)}\binom{n-j}{l}\binom{t+\alpha-1}{w}\left(\frac{\theta^s}{w+\beta-\frac{1}{k}}\right),$$

$$A_2 = \sum_{l=0}^{\infty}\sum_{t=0}^{\infty} d_{j-c+l-1,t}(-1)^{l+w} \frac{1}{B(\alpha,\beta)}\binom{n-j}{l}\binom{t+\alpha-1}{w}\left(\frac{\theta^{k(w+\beta)+s}}{w+\beta-\frac{1}{k}}\right).$$

Substituting P_x in (4.10) we may write

$$I_x = \frac{1}{(j-c)\binom{n-c}{j-c}}\mu^{(s)}_{j-c:n-c} - A_1 - \sum_{w=0}^{\infty} A_2 x^{-k(w+\beta)+s},$$

Substituting P_x in (4.9) we may write

$$\begin{aligned}
\mu^{(r,s)}_{i,j:n} &= \sum_{m=0}^{j-i-1}(-1)^m\binom{j-i-1}{m}c_{i,j:n} \\
&\qquad \int_{\theta}^{\infty} x^r F(x)^{c-1} f(x)\left(\frac{(n-j)!}{(n-c)!}\mu^{(s)}_{j-c:n-c} - A_1 - \sum_{w=0}^{\infty} A_2 x^{-k(w+\beta)+s}\right)dx \\
&= \sum_{m=0}^{j-i-1}(-1)^m\binom{j-i-1}{m}c_{i,j:n} \\
&\qquad \left\{\left(\frac{(n-j)!}{(n-c)!}\mu^{(s)}_{j-c:n-c} - A_1\right)\frac{1}{c}\mu^{(r)}_{c:c} - \sum_{w=0}^{\infty} A_2\int_{\theta}^{\infty} x^{-k(w+\beta)+s+r}F(x)^{c-1}f(x)dx\right\} \\
&= \sum_{m=0}^{j-i-1}(-1)^m\binom{j-i-1}{m}c_{i,j:n}\frac{1}{c}\left\{\left(\frac{(n-j)!}{(n-c)!}\mu^{(s)}_{j-c:n-c} - A_1\right)\mu^{(r)}_{c:c} - \sum_{w=0}^{\infty} A_2\mu^{(-k(w+\beta)+s+r)}_{c:c}\right\}.
\end{aligned}$$

TABLE 2. The product moments of BP order statistics for n up to 4, $\alpha = 0.5$, $\beta = 2$, $k = 2$ and $\theta = 3$.

n	j	i	s	$r=1$	$r=2$	$r=3$	$r=4$
2	2	1	1	12.4912	40.4999	133.7166	454.8545
2	2	1	2	54.9190	182.2238	624.0422	2272.4535
2	2	1	3	312.5315	1080.3774	3985.4485	17288.2410
2	2	1	4	3938.8957	14509.0304	61154.9152	382519.9625
3	2	1	1	10.3194	32.0382	99.8308	312.5048
3	2	1	2	35.0581	109.3500	342.7013	1080.7330
3	2	1	3	121.7008	381.9390	1206.4640	3845.1283
3	2	1	4	435.8676	1379.6420	4408.1661	14281.0160
3	3	1	1	12.9573	40.2577	125.5480	393.3948
3	3	1	2	61.2221	191.3330	601.0117	1900.5440
3	3	1	3	378.5782	1192.9711	3786.7812	12141.5237
3	3	1	4	5160.2986	16420.3945	52788.9373	172245.0006
3	3	2	1	14.1969	49.2038	175.7710	658.6639
3	3	2	2	68.4769	245.9883	928.4138	3836.0836
3	3	2	3	437.3155	1666.2221	6963.1001	35878.0702
3	3	2	4	6220.5207	25727.0523	126267.6387	961033.8719
4	2	1	1	9.7367	29.7958	91.2958	280.1467
4	2	1	2	31.2157	95.6813	293.7208	903.2548
4	2	1	3	100.7065	309.2845	951.5849	2934.1159
4	2	1	4	327.3998	1007.8806	3109.6348	9620.3496
4	3	1	1	10.6463	32.5883	99.8822	306.5961
4	3	1	2	37.8800	116.1844	356.9181	1098.4792
4	3	1	3	138.4453	425.6921	1311.4732	4049.8120
4	3	1	4	525.5600	1621.0994	5012.7298	15547.3269
4	4	1	1	13.5081	41.3595	126.8038	389.3640
4	4	1	2	67.7426	207.9231	639.2339	1969.0587
4	4	1	3	448.8547	1382.0138	4264.1757	13190.3215
4	4	1	4	6539.8909	20203.5891	62579.8450	194464.7141
4	3	2	1	11.1577	35.9730	116.8493	383.1299
4	3	2	2	39.9208	129.8531	426.4453	1417.9431
4	3	2	3	146.9449	483.4947	1611.2129	5462.4692
4	3	2	4	563.1109	1881.7074	6400.6649	22336.0379
4	4	2	1	14.1669	45.7236	148.7023	488.2552
4	4	2	2	71.5230	233.3012	768.6606	2565.5788
4	4	2	3	478.1582	1582.1645	5307.2993	18135.6365
4	4	2	4	7035.8522	23653.2996	80983.3271	284503.2420
4	4	3	1	15.7315	57.5594	218.7661	881.6351
4	4	3	2	81.2318	310.3994	1259.2746	5680.4060
4	4	3	3	562.0795	2299.6144	10466.9439	59957.0873
4	4	3	4	8641.5601	38686.6018	208843.2790	1768648.0942

The value of $\mu_{i,j:n}^{(r,s)}$ are presented in Table 2 and 3, for n up to 5, r and s up to 4, $\alpha = 0.5$, $\beta = 2$, $k = 2$ and $\theta = 3$.

TABLE 3. The product moments of BP order statistics for n=5, $\alpha = 0.5$, $\beta = 2$, $k = 2$ and $\theta = 3$.

n	j	i	s	$r=1$	$r=2$	$r=3$	$r=4$
5	2	1	1	9.4794	28.8131	87.6293	266.6767
5	2	1	2	29.6657	90.2374	274.6636	836.6231
5	2	1	3	93.0828	283.3799	863.3538	2632.5017
5	2	1	4	292.9514	892.7181	2722.7067	8311.9844
5	3	1	1	9.9743	30.3215	92.2298	280.7197
5	3	1	2	33.0113	100.4446	305.8308	931.8808
5	3	1	3	110.2220	335.7310	1023.4148	3122.4197
5	3	1	4	371.9781	1134.4348	3462.8790	10581.4397
5	4	1	1	10.9843	33.3959	101.5948	309.2682
5	4	1	2	40.6811	123.8176	377.1130	1149.4648
5	4	1	3	155.3900	473.5526	1444.3349	4409.2730
5	4	1	4	619.6356	1891.2751	5778.2870	17673.7509
5	5	1	1	14.0325	42.6689	129.8224	395.2547
5	5	1	2	73.8878	224.9565	685.3854	2089.8643
5	5	1	3	517.3573	1577.5761	4814.6576	14708.3536
5	5	1	4	7937.1491	24240.4163	74107.2192	226823.2522
5	3	2	1	10.2710	32.2182	101.3613	319.9837
5	3	2	2	34.0704	107.2494	338.7825	1074.5238
5	3	2	3	114.0618	360.5521	1144.4483	3650.6543
5	3	2	4	386.1664	1226.8140	3917.1747	12584.3532
5	4	2	1	11.3144	35.5068	111.7617	353.0043
5	4	2	2	42.0152	132.3975	418.7031	1329.7047
5	4	2	3	161.0027	509.8935	1621.8658	5185.6739
5	4	2	4	644.5727	2054.0769	6581.3161	21226.2515
5	5	2	1	14.4585	45.3947	142.9571	451.7877
5	5	2	2	76.3686	240.9287	762.9003	2426.2359
5	5	2	3	536.8118	1703.7889	5432.5733	17417.2820
5	5	2	4	8268.3726	26405.4245	84799.2887	274180.0887
5	4	3	1	11.9661	39.9609	134.8812	461.3389
5	4	3	2	44.7241	151.1847	517.9791	1805.4815
5	4	3	3	172.7990	593.2380	2072.6508	7412.6813
5	4	3	4	699.3245	2450.4160	8793.8290	32642.6146
5	5	3	1	15.3014	51.1608	172.9178	592.3481
5	5	3	2	81.4313	276.1256	949.3996	3322.8592
5	5	3	3	578.0823	1996.6757	7024.7400	25328.9711
5	5	3	4	8998.9712	31700.7848	114368.2996	426464.8163
5	5	4	1	17.1300	65.5584	262.0106	1118.1629
5	5	4	2	93.3346	374.8956	1609.6647	7757.8963
5	5	4	3	686.5053	2969.3861	14412.4423	89014.5938
5	5	4	4	11169.8350	53093.9363	307018.0671	2794711.0368

5. Conclusion

We studied the certain recurrence relations for the single and product moments of the order statistics of a random sample of size n arising from beta-Pareto distribution are derived. It would be interest to a numerical application to illustrate the usefulness of the result in future work. The addition of this section was the kind suggestion of an anonymous referee of the journal.

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