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BOTTOM-k-LISTS

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Abstract: Consider an infinite sequence of iid continuous random variables. In this paper, we introduce duals of top-k-lists which is called "Bottom-k-lists". We also derive the joint probability density function (pdf) of the ordered elements of ℓth bottom-k-list and its relation with the joint pdf of dual (lower) generalized order statistics (dGOS). In addition, we give a distributional relation between each element of ℓth bottom-k-list, ordinary order statistics (oOS), and lower k-records.

Key words: Lower k-records, Bottom-k-lists, Dual generalized order statistics, Meijer's G function, Ordinary order statistics.
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1. Introduction

Top-k-lists are used in many areas such as sports, meteorology etc. The distribution theory of top-k-lists based on any iid continuous random sequence was studied by López-Blázques and Wesołowski (2007), and Tanil (2009). In this paper we introduce the concept of "Bottom-k-lists" which is the decreasingly ordered analogue to the "Top-k-lists" defined by López-Blázquez and Wesolowski (2007). One possible application of Bottom-k-lists can be in the analysis of athletizm scores. For example, best $k=3$ scores of a 100 meter athlet are obviously his/her lowest 3 run times and by knowing the distributions of these run times up to date, we can qualify and interpret the performance of this sportsman for the future.

The concept of lower k-records plays a very important role in order to obtain bottom-k-lists. A bottom-k-list alters when a new lower k-record value appears. At this moment, the largest element of the list is removed and the observation, which reveals the new lower k-record, enters the list. Now, we recollect the definition of lower k-records. Formally, let $X_{i:m}$ denote the ith oOS from a random sample of size m. Let $\mathbf{X}=\{X_i\}_{i\in\mathbb{N}^+}$ be an infinite sequence of iid continuous random variables with distribution function (df) F and corresponding pdf f. In this sequence, the lower k-record times defined recurrently as

$$L_k(0)=1,$$

$$L_k(\ell+1)=\min\left\{j>L_k(\ell):X_{k:j+k-1}<X_{k:L_k(\ell)+k-1}\right\},\ell\geq 0.$$

The sequence $\left\{Z_\ell^{(k)},\ell\geq 0\right\}$ with $Z_\ell^{(k)}=X_{k:L_k(\ell)+k-1}$ is called the sequence of lower k-record values of $\mathbf{X}$. It is well known that the marginal pdf of $Z_\ell^{(k)}$ is given as follows (see Pawlas and Szynal, 1998): For $F^{-1}(0)<y<F^{-1}(1)$,

$$f^{Z_\ell^{(k)}}(y)=\frac{k^{\ell+1}}{\ell!}\left(-\ln F(y)\right)^\ell F(y)^{k-1}f(y),\ell\geq 0 \tag{1.1}$$

Now, we define the concept of bottom-k-lists as follows: The ℓth bottom-k-list from $\mathbf{X}$ is defined as

$$B_\ell^{(k)}=\left(W_{1,\ell}^{(k)},W_{2,\ell}^{(k)},...,W_{k,\ell}^{(k)}\right),\ell\geq 0$$

where $W_{j,\ell}^{(k)} = X_{j:L_k(\ell)+k-1}$, $j=1,2,...,k$, $\ell = 0,1,2,....$ Denote $W_{k,\ell}^{(k)} = Z_\ell^{(k)}$. Using the above definition, it is easy to rewrite the bottom-k-lists as

$$B_0^{(k)} = (X_{1:k}, X_{2:k}, ..., X_{k:k})$$

and for $\ell > 0$

$$B_\ell^{(k)} = \text{ord}\left(W_{1,\ell-1}^{(k)}, W_{2,\ell-1}^{(k)}, ..., W_{k-1,\ell-1}^{(k)}, X_{L_k(\ell)}\right)$$

where "ord" is a function which arranges elements of a given vector in increasing order. In order to clarify bottom-k-lists, let us consider the following sequence of observations: $3.0, 3.1, 2.8, 2.7, 1.5, 3.6, 0.5, 2.9, 2.1, 3.9, 2.5, ...$ For $k=3$, the current lower k-records and the current bottom-k-lists from the above sequence are as follows:

Time index of data	1	2	3	4	5	6	7	8	9	10	11
Data	3.0	3.1	2.8	2.7	1.5	3.6	0.5	2.9	2.1	3.9	2.5
Current lower 3- record	–	–	3.1	3.0	2.8	2.8	2.7	2.7	2.1	2.1	2.1
Current bottom-3- list	–	–	$B_0^{(3)}$	$B_1^{(3)}$	$B_2^{(3)}$	$B_2^{(3)}$	$B_3^{(3)}$	$B_3^{(3)}$	$B_4^{(3)}$	$B_4^{(3)}$	$B_4^{(3)}$

where $B_0^{(3)} = (2.8, 3.0, 3.1)$, $B_1^{(3)} = (2.7, 2.8, 3.0)$, $B_2^{(3)} = (1.5, 2.7, 2.8)$, $B_3^{(3)} = (0.5, 1.5, 2.7)$, and $B_4^{(3)} = (0.5, 1.5, 2.1)$.

In addition, the probability of the ℓth change of the list is equal to the probability of the event $\left\{X_{L_k(\ell-1)+i} < W_{k,\ell-1}^{(k)}\right\}$, where $X_{L_k(\ell-1)+i}$ is ith new observation from the sequence after $(\ell-1)$th change of the list for $\ell, i \in \mathbb{N}^+$. Then, using (1.1), one can easily calculate this probability as in the following:

$$\begin{aligned} P\left\{X_{L_k(\ell-1)+i} < W_{k,\ell-1}^{(k)}\right\} &= \int_{-\infty}^{\infty}\int_{-\infty}^{y} f(x)\, f^{W_{k,\ell-1}^{(k)}}(y)\, dx dy \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{y} f(x)\, f^{Z_{\ell-1}^{(k)}}(y)\, dx dy \\ &= \frac{k^\ell}{(\ell-1)!}\int_0^1 (-\ln u)^{\ell-1} u^k du \\ &= \left(\frac{k}{k+1}\right)^\ell \end{aligned}$$

which is the same with the probability of changing the top-k-list (see Tanil, 2009, Theorem 4.1, p. 2194).

In this paper, we give some definitions and auxiliary results in section 2. After that, in section 3, we obtain the joint pdf of ℓth bottom-k-list and say that this list is related to dGOS. Finally, we show that the marginal pdf of each element of bottom-k-list is equal to a linear combination of the pdf's of oOS and the pdf's of lower k-records.

2. Some Definitions and Auxiliary Results

Definition 1. (see Pawlas and Szynal, 2001) Let F be an absolutely continuous df with pdf f. The random variables $X(1,n,\tilde{m},k') > X(2,n,\tilde{m},k') > ... > X(n,n,\tilde{m},k')$ are called "dual generalized order statistics" based on F if their joint pdf is given by

$$f_{X(1,n,\tilde{m},k'),...,X(n,n,\tilde{m},k')}(v_1, v_2, ..., v_n) = k' F(v_n)^{k'-1} f(v_n) \prod_{i=1}^{n-1} \gamma_i F(v_i)^{m_i} f(v_i),$$

on the cone $F^{-1}(0) < v_n \leq v_{n-1} \leq ... \leq v_1 < F^{-1}(1)$ of $\mathbb{R}^n$ with parameters $n \in \{1,2,3,4,...\}$, $k^{'} > 0$, $\tilde{m} = (m_1, m_2, ..., m_{n-1}) \in \mathbb{N}^{n-1}$, $M_r = \sum_{i=r}^{n-1} m_i$, such that $\gamma_r = k^{'} + n - r + M_r > 0$ for all $r \in \{1,2,...,n-1\}$. Denote $\gamma_n = k^{'}$.

Definition 2. (see Erdelyi et al., 1953; Luke, 1969; Mathai, 1993; Mathai and Haubold, 2008, p. 50) For positive integer t, let $G^{t,0}_{t,t}$ be an existing particular Meijer's G function which is defined as

$$G^{t,0}_{t,t}\left[z\,|\gamma_1, ..., \gamma_t\right] = \frac{1}{2\pi i}\int_L \frac{z^{-s}}{\prod_{j=1}^{t}(\gamma_j - 1 + s)} ds$$

where L is an appropriately chosen integration path, $i = \sqrt{-1}$, $|z| < 1$, and $s \in \mathbb{C}$.

Lemma 1. *(see Burkschat et al., 2003) For $1 \leq t \leq n$, the pdf of t-th dGOS is*

$$f^{X(t,n,\tilde{m},k^{'})}(v) = \left(\prod_{j=1}^{t}\gamma_j\right) G^{t,0}_{t,t}\left[F(v)\,|\gamma_1, ..., \gamma_t\right] f(v), \ v \in \mathbb{R}.$$

Lemma 2. *(see Burkschat and Lenz, 2009, p. 2098) It is true that*

$$G^{t,0}_{t,t}\left[z\,|k, k-1, ..., k-t+1\right] = \frac{1}{(t-1)!} z^{k-t}(1-z)^{t-1}$$

where $k, t \in \{1,2,3,...\}$, $t \leq k$, and $z \in (0,1)$.

3. Model of Bottom-k-List and Their Relation with dGOS

We give the joint pdf of the elements of ℓth bottom-k-list with the following theorem:

Theorem 1. *For $\ell \geq 0$ and $k \geq 1$, the joint pdf of $B^{(k)}_\ell$ is*

$$f^{B^{(k)}_\ell}(y_1, y_2, ..., y_k) = \frac{k! k^\ell}{\ell!}\left(-\ln F(y_k)\right)^\ell \prod_{i=1}^{k} f(y_i), \ \text{for } y_1 < y_2 < ... < y_k \quad (3.1)$$

Proof. The conditional joint pdf of a random sample $W^{(k)}_{1,\ell}, W^{(k)}_{2,\ell}, ..., W^{(k)}_{k-1,\ell}$ given $W^{(k)}_{k,\ell} = y_k$ is the same structure as the joint pdf of oOS based on a sample of size $k-1$ with the pdf $f(y)/F(y_k)$, $y \leq y_k$. It is clear that the pdf in (3.1) can be obtained with the product of the pdf in (1.1) and the conditional joint pdf.

In this theorem, we note that (i) we obtain the joint pdf of oOS based on first k observations of $\mathbf{X}$ for $\ell = 0$, (ii) we obtain the marginal pdf of ℓth lower record value for $k = 1$. In addition, the decreasingly ordered random vector of $\left(W^{(k)}_{k,0}, W^{(k)}_{k,1}, ..., W^{(k)}_{k,\ell-1}, W^{(k)}_{k,\ell}, W^{(k)}_{k-1,\ell}, ..., W^{(k)}_{1,\ell}\right)$ is a particular submodel of dGOS $X(1, k+\ell, \widetilde{\mathbf{m}}, k') > X(2, k+\ell, \widetilde{\mathbf{m}}, k') > ... > X(k+\ell, k+\ell, \widetilde{\mathbf{m}}, k')$ with parameters

$$\widetilde{\mathbf{m}} = \tilde{m}^* = \left(\overbrace{-1, -1, ..., -1}^{\ell \text{ times}}, 0, ..., 0\right)$$

and $k' = 1$. Under these parameters, one can obtain the joint pdf in (3.1) by integrating the joint pdf of dGOS in definition 1 with respect to $v_1, v_2, ..., v_\ell$. It is clear that ℓth bottom-k-list is marginal of $X(1, k+\ell, \tilde{m}^*, 1), X(2, k+\ell, \tilde{m}^*, 1), ..., X(k+\ell, k+\ell, \tilde{m}^*, 1)$. In other words, for $\ell \geq 0$, $k \geq 1$, *and* $1 \leq r \leq k$, $W^{(k)}_{r,\ell} \overset{d}{=} X(\ell+k-r+1, k+\ell, \tilde{m}^*, 1)$ where $\overset{d}{=}$ means identical in distribution. Thus, from lemma 1, the marginal pdf of $W_{r,\ell}^{(k)}$ which is the $(k-r+1)$th largest value of ℓth bottom-k-list can be written as

$$f^{W_{r,\ell}^{(k)}}(y)=\frac{k!k^{\ell}}{(r-1)!}G_{\ell+k-r+1,\ell+k-r+1}^{\ell+k-r+1,0}\left[F(y)\left|\overbrace{k,...,k}^{\ell+1\text{ times}},k-1,k-2,...,r\right.\right]f(y) \qquad (3.2)$$

where $F^{-1}(0)<y<F^{-1}(1)$. Denote the marginal pdf of $W_{k,\ell}^{(k)}\left(=Z_n^{(k)}\right)$ is given in (1.1). Now, we give the following theorem which will be used for the particular Meijer's G function within formula (3.2) for $r<k$.

THEOREM 2. *For $\ell\geq 0$ and $1\leq r<k$, it is true that*

$$\begin{aligned}
&G_{\ell+k-r+1,\ell+k-r+1}^{\ell+k-r+1,0}\left[F(y)\left|\overbrace{k,...,k}^{\ell+1\text{ times}},k-1,...,r\right.\right]\\
&=\sum_{u=0}^{k-r-1}\frac{(-1)^{r-k+1+u}F(y)^{k-2-u}}{(u+1)^{\ell+1}u!(k-r-1-u)!}\\
&+\sum_{u=0}^{\ell}\frac{(-1)^{\ell-u}F(y)^{k-1}(-\ln F(y))^{u}}{u!(r-k)^{1-u+\ell}}\\
&\times\left(\sum_{u_1=0}^{(\ell-u)_1}\sum_{u_2=0}^{(\ell-u)_2}\cdots\sum_{u_{k-r-1}=0}^{(\ell-u)_{k-r-1}}\frac{\prod_{j=1}^{k-r-1}(-j)^{-u_j-1}}{(r-k)^{-\left(u_1+...+u_{k-r-1}\right)}}\right)^{\min\{1,k-r-1\}}
\end{aligned}$$

where $(\ell-u)_t=\ell-u-\sum_{v=0}^{t-1}u_v$ and $u_0=0$.

Proof. For $\ell\geq 0$ *and* $1\leq r<k$, from Definition 2, one can write

$$G_{\ell+k-r+1,\ell+k-r+1}^{\ell+k-r+1,0}\left[F(y)|k,...,k,k-1,...,r\right]=\frac{1}{2\pi i}\int_L\frac{F(y)^{-s}}{(k-1+s)^{\ell+1}(k-2+s)(k-3+s)...(r-1+s)}ds.$$

It is well known that the preceding integral can be evaluated by summing up the residues at simple poles of $(2-k)$, $(3-k)$,..., $(1-r)$, and $(\ell+1)\,th$ order pole of $(1-k)$. Let R_1 be the summation of the residues at the simple poles and R_2 be the residue of the $(\ell+1)\,th$ order pole. Using the residue theorem, we can write

$$R_1=\sum_{u=0}^{k-r-1}\lim_{s\to u+2-k}\frac{(s-(u+2-k))F(y)^{-s}}{(k-1+s)^{\ell+1}(k-2+s)(k-3+s)...(r-1+s)}$$

and

$$\begin{aligned}
R_2&=\frac{1}{\ell!}\lim_{s\to 1-k}\frac{d^{\ell}}{ds^{\ell}}\left(\frac{F(y)^{-s}}{(k-2+s)(k-3+s)...(r-1+s)}\right)\\
&=\frac{1}{\ell!}\lim_{s\to 1-k}\sum_{u=0}^{\ell}\binom{\ell}{u}\left(\frac{d^{u}}{ds^{u}}F(y)^{-s}\right)\left(\frac{d^{\ell-u}}{ds^{\ell-u}}\frac{1}{(k-2+s)(k-3+s)...(r-1+s)}\right).
\end{aligned}$$

Thus, one can easily obtain

$$R_1=\sum_{u=0}^{k-r-1}\frac{(-1)^{r-k+1+u}F(y)^{k-2-u}}{(u+1)^{\ell+1}u!(k-r-1-u)!},$$

and

$$R_2 = \sum_{u=0}^{\ell} \frac{(-1)^{\ell-u} F(y)^{k-1} (-\ln F(y))^u}{u!\,(r-k)^{1-u+\ell}} \times \left(\sum_{u_1=0}^{(\ell-u)_1} \sum_{u_2=0}^{(\ell-u)_2} \cdots \sum_{u_{k-r-1}=0}^{(\ell-u)_{k-r-1}} \frac{\prod_{j=1}^{k-r-1} (-j)^{-u_j-1}}{(r-k)^{-(u_1+\ldots+u_{k-r-1})}} \right)^{\min\{1,k-r-1\}}$$

where $(\ell-u)_t = \ell - u - \sum_{v=0}^{t-1} u_v$ and $u_0 = 0$. The Meijer's G function is equal to $R_1 + R_2$ so theorem is proved.

Corollary 1. *For $\ell = 0$, theorem 2 has a simple form as follows:*

$$G_{k-r+1,k-r+1}^{k-r+1,0}[F(y)\,|k,k-1,...,r] = \sum_{u=0}^{k-r-1} \frac{(-1)^{r-k+1+u} F(y)^{k-2-u}}{(u+1)!\,(k-r-1-u)!} + \frac{F(y)^{k-1}}{(r-k)} \left(\sum_{u_1=0}^{0} \sum_{u_2=0}^{0} \cdots \sum_{u_{k-r-1}=0}^{0} \frac{\prod_{j=1}^{k-r-1} (-j)^{-u_j-1}}{(r-k)^{-(u_1+\ldots+u_{k-r-1})}} \right)^{\min\{1,k-r-1\}}$$
$$= \sum_{u=0}^{k-r-1} \frac{(-1)^{r-k+1+u} F(y)^{k-2-u}}{(u+1)!\,(k-r-1-u)!} + \frac{F(y)^{k-1}}{(r-k)} \left(\frac{(-1)^{k-r-1}}{(k-r-1)!} \right)^{\min\{1,k-r-1\}}$$

(i) For $r = k-1$, then one has

$$G_{2,2}^{2,0}[F(y)\,|k,k-1] = \sum_{u=0}^{0} \frac{(-1)^u F(y)^{k-2-u}}{(u+1)!\,(-u)!} + \frac{F(y)^{k-1}}{-1} \left(\frac{(-1)^0}{0!} \right)^0 = F(y)^{k-2} (1 - F(y)),$$

and
(ii) For $r < k-1$, then one can write

$$G_{k-r+1,k-r+1}^{k-r+1,0}[F(y)\,|k,k-1,...,r] = \sum_{u=0}^{k-r-1} \frac{(-1)^{r-k+1+u} F(y)^{k-2-u}}{(u+1)!\,(k-r-1-u)!} + \frac{F(y)^{k-1}}{(r-k)} \left(\frac{(-1)^{k-r-1}}{(k-r-1)!} \right)^1$$
$$= \sum_{u=0}^{k-r-1} \frac{(-1)^{r-k+1+u} F(y)^{k-2-u}}{(u+1)!\,(k-r-1-u)!} + \frac{(-1)^{k-r} F(y)^{k-1}}{(k-r)!}$$
$$= \sum_{u=-1}^{k-r-1} \frac{(-1)^{r-k+1+u} F(y)^{k-2-u}}{(u+1)!\,(k-r-1-u)!}$$
$$= \frac{F(y)^{r-1} (1-F(y))^{k-r}}{(k-r)!}$$

which agrees Lemma 2.

4. A Linear Combination

If we write the polynomial expansion of particular Meijer's G function in theorem 2 into equation (3.2), then it is easy to see that we can obtain the marginal pdf of $W_{r,\ell}^{(k)}$ as the following form: For $\ell \geq 0$ and $1 \leq r < k$,

$$f^{W_{r,\ell}^{(k)}}(y) = \sum_{u=0}^{k-r-1} \Psi_1^{(k,\ell,r,u)} f^{W_{k-1-u,0}^{(k-1-u)}}(y) + \sum_{u=0}^{\ell} \Psi_2^{(k,\ell,r,u)} f^{W_{k,u}^{(k)}}(y) \tag{4.1}$$

where

$$\Psi_1^{(k,\ell,r,u)} = \frac{k!k^{\ell}\,(-1)^{r-k+1+u}}{(r-1)!\,(u+1)^{\ell+1}\,u!\,(k-r-1-u)!\,(k-1-u)}$$

and

$$\Psi_2^{(k,\ell,r,u)} = \frac{(-1)^{\ell-u}\,(k-1)!k^{\ell-u}}{(r-1)!\,(r-k)^{1-u+\ell}} \left(\sum_{u_1=0}^{(\ell-u)_1} \sum_{u_2=0}^{(\ell-u)_2} \cdots \sum_{u_{k-r-1}=0}^{(\ell-u)_{k-r-1}} \frac{\prod_{j=1}^{k-r-1} (-j)^{-u_j-1}}{(r-k)^{-(u_1+\ldots+u_{k-r-1})}} \right)^{\min\{1,k-r-1\}} .$$

Note that $W_{k-1-u,0}^{(k-1-u)}$ and $W_{k,u}^{(k)}$ in (4.1) are *maximum oOS* and *uth lower k-record value*, respectively. In other words, the marginal pdf of $W_{r,\ell}^{(k)}$ is equal to the linear combination of the pdf's of oOS and the pdf's of lower k-records as formula (4.1). In order to see advantages of usage of (4.1), we give the following example:

Example 1. Let $X_1, X_2, X_3, \ldots$ be an infinite sequence of independent uniform$(0,1)$ distributed random variables. For $\ell \geq 0$ and $1 \leq r < k$, from (4.1) the expected value of $W_{r,\ell}^{(k)}$ is written as

$$\begin{aligned} E\left(W_{r,\ell}^{(k)}\right) &= \sum_{u=0}^{k-r-1} \Psi_1^{(k,\ell,r,u)} E\left(W_{k-1-u,0}^{(k-1-u)}\right) \\ &+ \sum_{u=0}^{\ell} \Psi_2^{(k,\ell,r,u)} E\left(W_{k,u}^{(k)}\right). \end{aligned}$$

In the statistical literature, it is well known that $E\left(W_{k-1-u,0}^{(k-1-u)}\right) = \frac{k-1-u}{k-u}$ and $E\left(W_{k,u}^{(k)}\right) = \left(\frac{k}{k+1}\right)^{u+1}$ under uniform$(0,1)$ distribution. Thus, one can easily write

$$\begin{aligned} E\left(W_{r,\ell}^{(k)}\right) &= \sum_{u=0}^{k-r-1} \Psi_1^{(k,\ell,r,u)} \frac{k-1-u}{k-u} \\ &+ \sum_{u=0}^{\ell} \Psi_2^{(k,\ell,r,u)} \left(\frac{k}{k+1}\right)^{u+1}. \end{aligned}$$

5. Conclusions

In this paper, we introduce a new concept of bottom-k-lists which are the decreasingly ordered analogue to top-k-lists and derive the joint pdf's of bottom-k-lists. Also, the marginal pdf of each element of bottom-k-list is showed as the linear combination of the pdf's of oOS and the pdf's of lower k-records. In the statistical literature, there are many studies about oOS and lower k-records. Using formula (4.1) and the results of these studies, many new results about the each element of ℓth bottom-k-list are easily obtained such as single moments, characteristic functions, moment generating functions, etc.

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