DOI: 10.34056/aujef.1344384

Investigating Middle School Students' Problem Solving Approaches and Geometric Habits of Mind

Ortaokul Öğrencilerinin Geometri Problemlerini Çözüm Yaklaşımlarının ve Geometrik Düşünme Alışkanlıklarının İncelenmesi

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Article Type: Research Article

Application Date: 16.08.2023 **Accepted Date:** 10.10.2023

To Cite This Article: Özen Ünal, D., & Ürün, Ö. (2023). Ortaokul öğrencilerinin geometri problemlerini çözüm yaklaşımlarının ve geometrik düşünme alışkanlıklarının incelenmesi. *Anadolu University Journal of Education Faculty (AUJEF), 7*(4), 765-782.

ABSTRACT: This study aimed to determine the geometry problem solving processes and geometric habits of mind of the middle school students. Within the qualitative inquiry, case study design was implemented. The participants consisted of 4 eighth grade students enrolled in a public school. Three open-ended questions that have the potential to reveal geometric habits of mind were used and clinical interviews were conducted. The solution approaches of the students were determined and then analyzed using Geometric Habits of Mind (GHoM by Driscoll et al., 2007) framework by the stages of content analysis. The solution approaches of the students for each problem were analyzed and codes and themes were defined in line with the theoretical framework. This study indicated that the students were able to reason in various ways in the context of the components and indicators of the GHoM. In addition, while students' geometric thinking examined in accordance with the components, it is clearly seen that the students have adequate geometric habits of mind.

Keywords: Geometric habits of mind, geometric thinking, problem solving, middle school students

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ÖZ: Bu çalışmada ortaokul öğrencilerinin geometri problemlerini çözme süreçleri ve geometrik düşünme alışkanlıklarının belirlenmesi amaçlanmıştır. Araştırma deseni olarak nitel araştırma yöntemlerinden durum çalışması deseni benimsenmiştir. Katılımcılar bir devlet okulunda öğrenim gören 4 sekizinci sınıf öğrencisidir. Araştırmada geometrik düşünme alışkanlıklarını ortaya çıkarma potansiyeline sahip üç açık uçlu soru ve klinik görüşme kayıtlarından yararlanılmıştır. Öğrencilerin problem çözme yaklaşımları belirlenmiş, daha sonra geometrik düşünme alışkanlıkları (Driscoll et al., 2007) kuramsal çatısının bileşenleri doğrultusunda içerik analizinin gerçekleştirilmiştir. Öğrencilerin her bir probleme ait çözüm yaklaşımları araştırmacı tarafından incelenmiş ve geometrik düşünme alışkanlıkları bileşenleri doğrultusunda kodlar ve temalar oluşturulmuştur. Ortaokul öğrencilerinin geometri problemlerindeki düşünme yollarını ortaya koyan bu çalışmada öğrencilerin geometrik düşünme alışkanlıklarının bileşenleri ve göstergeleri bağlamında çeşitli yollarla muhakeme edebildikleri görülmüştür. Ayrıca her bir alışkanlık incelendiğinde öğrencilerin geometrik düşünme alışkanlıklarına yeterli derecede sahip oldukları söylenebilir.

Anahtar sözcükler: Geometrik düşünme alışkanlıkları, geometrik düşünme, problem çözme, ortaokul öğrencileri

1. INTRODUCTION

Mathematics is a universal language that has developed continuously throughout history and has connections to almost all disciplines. Along with all its other benefits, it makes a significant contribution to individuals' ability to produce solutions to the problems they encounter (Gözel, 2016). Problem solving ability is also essential for the survival of the human race. People with well-developed problem-solving skills will be able to use knowledge for survival effectively and overcome the challenges they face (Altun, 2016). Thus, in the current mathematics curriculum, it is intended to develop fundamental skills including reasoning, connection, and problem-solving as well as four operation skills. To accomplish the goal of enhancing problem solving skills of the students, instructional environments that will allow students to reflect on their ideas, reason, and justify their solutions should be designed. Students should solve problems through explorations, develop thoughts in this direction, and apply this in their daily lives by developing the habit of doing so (National Council of Teachers of Mathematics-NCTM, 2000). It has been found that students who actively engage in the problem-solving process have higher levels of motivation and success in the process (Thorson, 1999) because solving problems requires a collective function of a variety of skills, knowledge, beliefs, attitudes, intuitions, and prior acquisitions (Charles, Lester, & O'Daffer, 1987). An individual must go through a cognitive process when solving a problem, such as understanding the problem, selecting the data/information required for the solution, answering the problem, and determining whether the answer is logical or not (Charles, 1985). Therefore, problem solving is a process, not a result (Kneeland, 2001; Latterell, 2003). The process of attempting to solve a problem that one encounters is referred to as "problem solving process". Thornton (1998, p. 10) defined the problem-solving process as what people do when they do not know how to reach a goal.

Geometry plays a significant role in the improvement of problem-solving skills. People's search for answers to the problems brought on by daily needs forced them to do geometry. The ancient Greek mathematicians gave these answers, which were largely the result of trial and error, a theoretical character and they attempted to prove the solutions to the problems that existed at the time (Baki, 2015). In the process, geometry has been used to help solve many problems. By attempting to solve problems arising from daily needs, geometry also made significant contributions to the development of mathematics. Therefore, problems and the problem solving process have an important place in the development of mathematical thinking. The ability to abstract, predict, generalize, establish and test hypotheses, support and derive other propositions from the ones at hand, and thus arrive at new knowledge (Alkan ve Güzel, 2005) by utilizing prior mathematical knowledge and concepts has been attained by people with mathematical thinking. A set of habits of minds help the mathematical thinking process repeat and develop. According to Köse and Tanışlı (2014), habits of mind are a person's propensity to pick and use a strategy that can be successful in solving a problem. Geometric Habits of Mind (GHoM) and algebraic habits of mind (Driscoll, 1999; Driscoll, DiMatteo, Nikula, and Egan, 2007) are examples of mind habits that are prominent in geometry and algebra in the existing literature. In these studies, ways of thinking for individuals to become successful problem solvers were defined, and analysis of thinking evidence was included.

As a result of the study conducted with teachers and their students in grades 5 to 10 to enhance their geometric thinking, Driscoll et al. (2007) documented geometric habits of mind. Geometric Habits of Mind is a theoretical framework that emphasizes productive ways of thinking by focusing on geometric thinking and reveals these ways with four interconnected geometric habits of minds: reasoning with relationships, generalizing geometric ideas, investigating invariants, balancing exploration and reflection (Driscoll et al., 2008). The reasoning with relationships habit also includes ways of thinking in the form of using special reasoning skills such as proportion and symmetry while focusing on the relationships between geometric shapes and subfigures within a shape. The generalizing geometric ideas habit includes reflections on questions such as “Is this situation always valid”, and “why this situation is always valid?” regarding the situations that arise from geometric concepts and operations. Investigating invariants habit includes analyses that reveal which properties of a geometric shape are affected under a transformation. Finally, balancing exploration and reflection habit includes indications of the benefits of the information obtained by trying various ways in a problem solving situation and evaluating the situation regularly.

Individuals can hone their reasoning styles by practicing various habits, such as solving problems, thinking back on experiences, and applying (transferring) those insights from the experiences to new situations. In this context, students' habits of mind can be developed with problems that can reveal these habits. For this reason, it is of great importance to explore geometric habits of mind. As a result of a review of existing literature, studies on the geometry learning domain including individuals' knowledge of geometric concepts (volume of the cylinder) (Koç & Bozkurt, 2012), specification and generalization processes in geometry (Yıldırım, 2015), problem solving strategies in geometry lessons (Aydoğdu & Keşan, 2016), and the different strategies used in the process of solving multi-solution geometry problems (Yılmaz & Köse, 2015) were found. In addition, studies that examine the determination of individuals' geometric habits of minds and that focus on fostering geometric thinking were found (Altıkardeş & Yiğit-Koyunkaya, 2022; Jacobbe, T. & Millman, 2009; Köse & Tanışlı, 2014; Özen, 2015; Özen-Ünal & Köse, 2019; Özen-Ünal, Ulusan & Gürlek, 2022; Özüm-Bülbül & Güven, 2020; Tolga, 2017; Uygan, 2016, Ünveren-Bilgiç, 2018; Zunlu, 2022). Among these studies, Özen-Ünal, Ulusan, and Gürlek (2022) examined middle school students’ thinking ways and geometric habits of minds in the context of activities to develop geometric thinking. Uygan (2016), on the other hand, designed and conducted a teaching experiment that utilized a dynamic geometry environment to develop middle school students’ geometric habits of mind. There are very few studies examining students' approaches to solving geometry problems and the geometric habits of minds elicited during this process. It is anticipated that research in this area will contribute to future initiatives and studies that can be designed for the development of geometric thinking. In line with this idea, this study examined the approaches middle school students used to solve geometry problems and their geometric habits of mind.

2. METHOD

2.1. Design

This study focused on geometric problem solving approaches and middle school students’ geometric habits of mind. Qualitative research methods focusing on researching and understanding social phenomena in the environment they connect to, were used in the collection, analysis and interpretation of research data. As the research focused on examining the problem-solving processes of secondary school students in terms of geometric habits of mind, a case study (Yin, 2009) research design was used.

2.2. Participants

The study participants were four 8th grade students studying at a public middle school in Aydın. The study conducted in the 2018-2019 academic year spring season. The selection of the participating students determined by employing criterion sampling, which is one of the purposeful sampling techniques that allows for the in-depth study of the cases that are thought to have rich knowledge, the discovery, and explanation of facts and phenomenon (Yıldırım ve Şimşek, 2016). The study participants selected among volunteer 8^{th} grade students with high geometric thinking skills. The teachers shared their perspectives about students' difficulties related to geometric thinking skills. The students participating in the research were given the pseudonyms: S1, S2, S3, S4.

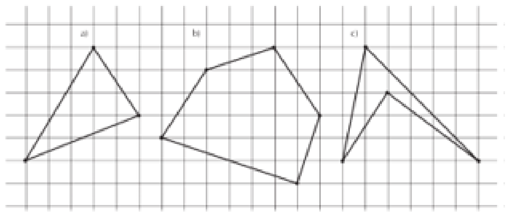
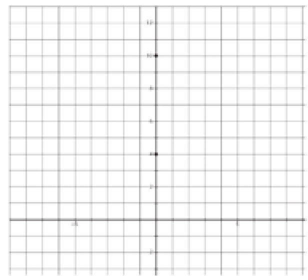
2.3. Data Collection

Worksheets with three open-ended problems were used in the study to gather data and reveal the students' habits of mind. Clinical interviews were used to collect the study's data (Goldin, 1998; Ginsburg, 1981; Hunting, 1997) because it allows researchers to examine cognitive processes and reveal in-depth student thought. The students were asked to thoroughly describe their solutions during the clinical interview. The interviews, which each lasted around 30 minutes, were captured on video. Students were instructed to think aloud throughout the recording. The problems in the worksheets were chosen from among those Driscoll et al. (2007) used in their projects for teachers and students in grades 5 through 10 to develop their geometric thinking.

For the first task, Investigating Area By Folding Paper by Driscoll et al., 2007, students were instructed to fold a square piece of paper into a new shape in a desired area and then to explain how they reasoned that the shape was equal to the given area. This task given in Table 1 consists of 5 stages. The second problem, Finding Area in Different Ways by Driscoll et al., 2007, consists of three stages that involve calculating the areas of three polygons given on a grid paper and evaluating the calculations to do so (Table 1, Task 2). The third task, Area of a Triangle by Driscoll et al., 2007, consists of determining all possible coordinates of the third vertex of a triangle, given the coordinates of the two vertices and area of the triangle, and six stages for special triangles (Table 1, Task 3).

The process of gathering the data was done without interfering with the students' scheduled classes and outside of those times. The interviews were conducted in a quiet classroom environment where the students could meet with the researcher in a one-on-one setting and feel comfortable. The students were asked to think aloud while answering the questions and to make detailed explanations about their solutions. The problem of Investigating Area by Paper Folding and the other two tasks were applied together on two separate days and at different times. The students were given enough time to come up with their solutions, and each clinical interview lasted approximately 30 minutes.

Table 1: *Tasks Used in Data Collection*

Task 1. INVESTIGATING AREA BY FOLDING PAPER	
For each problem, start with a square sheet of paper and make folds in the sheet of paper to construct a new shape, then explain how you know the shape you constructed has the specified area. 1. Construct a square with exactly $\frac{1}{4}$ the area of the original square. Explain how you know it has $\frac{1}{4}$ the area. 2. Construct a triangle with exactly $\frac{1}{4}$ the area of the original square. Explain how you know it has $\frac{1}{4}$ the area.	3. Construct another triangle, also with $\frac{1}{4}$ the area, which is not congruent to the first one you constructed. Explain how you know it has $\frac{1}{4}$ the area. 4. Construct a square with exactly $\frac{1}{2}$ the area of the original square. Explain how you know it has $\frac{1}{2}$ the area. 6. Construct another square, also with $\frac{1}{2}$ the area, which is oriented differently than the one you constructed in number 4. Explain how you know it has $\frac{1}{2}$ the area.
Task 2. FINDING AREA IN DIFFERENT WAYS	**Task 3. AREA OF THE TRIANGLE**
1. Here are three polygons:  a. Using any method you want, calculate the area of each polygon. b. Describe at least 3 methods for calculating the area of polygon b. c. Will these methods work to calculate the areas of polygons a and c? Explain why or why not.	Two vertices of a triangle are located at (0, 4) and (0, 10). The area of the triangle is 12 units^2.  a. What are all possible positions for the third vertex? b. Explain how you know these vertices create triangles with an area of 12 units^2. Write a convincing mathematical explanation. c. How do you know there aren't any more? d. How many right triangles are there? List the coordinates of the third vertex for each of the right triangles. e. How many isosceles triangles are there? List the coordinates of the third vertex for each of the isosceles triangles.

2.4. Data Analysis

First, the clinical interview data were transcribed. Second, the approaches that the students used were identified in the transcribed data, and they were subsequently using the stages of content analysis in line with the GHoM framework's components. The solution approaches of the students for each problem were examined by the researcher and coded using the framework. The solution approaches of the students and the habits of mind that emerged in this process were visualized through diagrams. The GHoM framework consists of four basic geometric habits of minds. These four habits of mind and the indicators of each step are summarized in Table 2 (Driscoll et al., 2007; Özen, 2015).

The data were coded using the open coding technique by two independent coders to ensure the reliability of the study. The opinions with a high percentage were determined to be the common opinion after discussing the disagreements between the two coders and reaching a consensus (Lincoln & Guba, 1985). The inter-coder reliability coefficient (Miles & Huberman, 2014) was found to be .93. To ensure the validity of the research, it was ensured that the students had enough time to work on the problem, member checking was used for the answers that were not understood/not clearly expressed by the students, and detaied rich descriptions and quotations were used in the analysis process.

Table 2: *Habits and Indicators in the Framework of Geometric Habits of Mind used in Data Analysis*

Conceptual Framework of the Geometric Habits of Mind

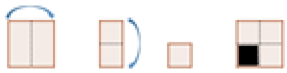
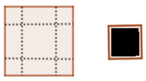
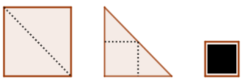
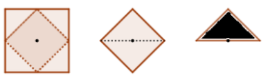
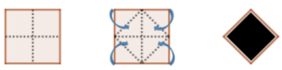
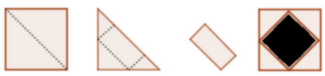
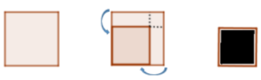
Reasoning with Relationships	*Focus on relationships among separate figures, by…* • comparing two or more geometric figures by enumerating some properties they have in common • contrasting two or more geometric figures by noting properties they do not have in common • comparing two or more geometric figures by considering relationships for their one-dimensional, two-dimensional, or three-dimensional components	*Focus on relationships among the pieces in a single figure, by…* • noticing and relating subfigures within a geometric figure • constructing subfigures within a geometric figure • relating two geometric figures by noticing they can be seen as parts of a single geometric figure	*Use spatial reasoning skills to focus on relationships, by…* • reasoning proportionally about two or more geometric figures • using symmetry to relate geometric figures
Generalizing Geometric Ideas	*Seek solutions from familiar cases or known solutions, by…* • considering relevant special cases (such as the side lengths which are a whole number) • looking beyond special cases to some other examples that fit (*Trying a side length which is not a whole number*) • generating new cases by changing features in cases already identified (*reflection, parallel displacement)* • intuiting that there are other solutions, without knowing how to generate them	*Seek a range of solutions using assumed simplifying conditions, by…* • recognizing that the given conditions work for an infinite set, but considering only a discrete set • seeing an infinite, continuously varying set of cases that work, but limiting the set or jumping to the wrong conclusion about the set, or • reaching an incorrect conclusion about the set	*Seek complete solution sets or general rules, by…* • seeing the entire set of solutions and explaining why there are no more • noticing a rule that is universally true for a class of geometric figures • situating problems or rules in broader contexts
Investigating Invariants	*Use dynamic thinking and searching, by…* • thinking dynamically about a static case • wondering about what changes and what stays the same when transformation is applied • generating a number of cases of transformation effects and looking for commonalities • thinking about the effects of moving a point or figure continuously and predicting occurrences in between one point and another • considering limit cases and extreme cases under transformations	*Check evidence of effects, by…* • intuiting that not everything is changing as a transformation is applied • noticing that the same effect appears to happen each time a particular type of transformation is applied • noticing invariants when a transformation is applied and explaining why there are invariants	
Balancing Exploration and Reflection	*Put exploration in the foreground by…* • drawing, playing, and/or exploring through intuition or guessing • drawing, playing, and/or exploring with regular stocktaking • considering previous similar situations • changing or considering changes to some feature of situation, condition, or geometric figure	*Put end goals in the foreground by…* • periodically returning to the big picture as a touchstone of progress • identifying intermediate steps that can help get to the goal • describing what the final state would look like • making reasoned conjectures about solutions, creating ways to test conjectures	

3. FINDINGS

This section presents the findings obtained from this study aiming to identify the solution approaches in the geometric problem solving process and emerging geometric habits of minds of the 8th grade students. While the geometric habits of minds that emerged as a result of the analyses were presented separately for each problem, the approaches used by the students were associated with the emerging habits of minds and supported by visual tools and direct quotations from clinical interviews.

For the first task, Investigating Area by Paper Folding, the students were asked to fold a square piece of paper in a desired area to create a new shape and explain how they understood that the formed shape was equal to the desired area. In this problem, students are expected to focus on the subfigures in a figure in the context of the reasoning with relationships habit, to use special reasoning skills and then to reach generalizations based on familiar situations and known results, and to make different folds continuously by emphasizing exploration in the context of the exploration and reflection component and to consider the information they have obtained by regularly evaluating the situation. Table 3 displays the solution approaches of the students and the habits of mind that emerged.

Table 3: *Students' Solution Approaches (Ways) and Habits of Minds in Investigating Area by Folding Task*

Stage	Strategies	S1	S2	S3	S4
1		-Reasoning with relationships	-Reasoning with relationships	-Reasoning with relationships	-Reasoning with relationships
		-Generalizing geometric ideas -Balancing exploration and reflection			
		-Generalizing geometric ideas -Balancing exploration and reflection			
2, 3		-Reasoning with relationships	-Reasoning with relationships	-Reasoning with relationships	-Reasoning with relationships
		-Generalizing geometric ideas -Balancing exploration and reflection			-Generalizing geometric ideas -Balancing exploration and reflection
			-Generalizing geometric ideas -Balancing exploration and reflection	-Generalizing geometric ideas -Balancing exploration and reflection	
4, 5		-Reasoning with relationships		-Generalizing geometric ideas -Balancing exploration and reflection	-Reasoning with relationships
			-Reasoning with relationships		
		-Generalizing geometric ideas -Balancing exploration and reflection		-Reasoning with relationships	-Generalizing geometric ideas* -Balancing exploration and reflection

S1 developed more than one strategy for the first stage of the problem and exhibited different habits of mind compared to other students. The students demonstrate the same habits of mind despite using various strategies in the other sub-questions. Yet, S2 could not develop a new strategy for the 5th stage of the problem. S4, on the other hand, acknowledged that she was unsure of how to approach the problem's fifth stage but sensed that there might be alternative solution approaches. Despite this, she was unable to provide a sufficient explanation. S4's drawings for the 4th and 5th stages of the problem are given in Figure 1. The student first came up with her solution for the fourth step of the problem by drawing it and then demonstrating it algebraically.

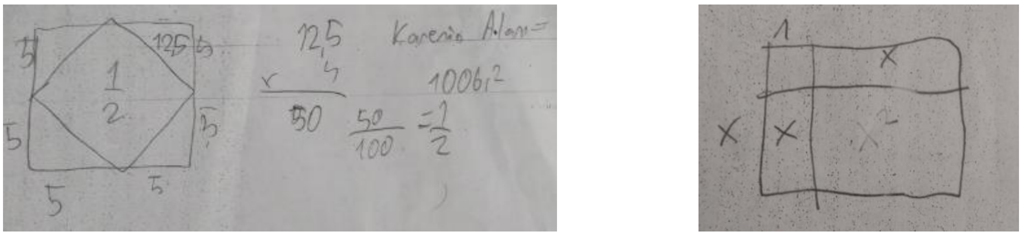


Figure 1: *Drawings of S4*

S4 made the following explanations for her drawing in the 5th step of the problem:

Researcher (R): Let our first squares side length be x.

S_4: This is x (pointing to the side of the square).

R: What should be the side length of the inner square? Since half of the area...

S_4: y...$\frac{x}{2}$...

R: if it is $\frac{x}{2}$ ...

S_4: Half of it.

R: What is the are of the first square?

S_4: x^2.

R: Yes. What is the half of it?

S_4: $\frac{x^2}{2}$.

R: Okay. What is the side lenght

S_4: Side lenght of $\frac{x^2}{2}$. We will take square root of it.

R: Okay.

S_4: $\frac{x^2}{\sqrt{2}}$

As seen in the explanations above, S4 brought the solution to the relevant step, but could not proceed further. The habits of mind and indicators of the students in the field research problem with paper folding are given in Figure 2.

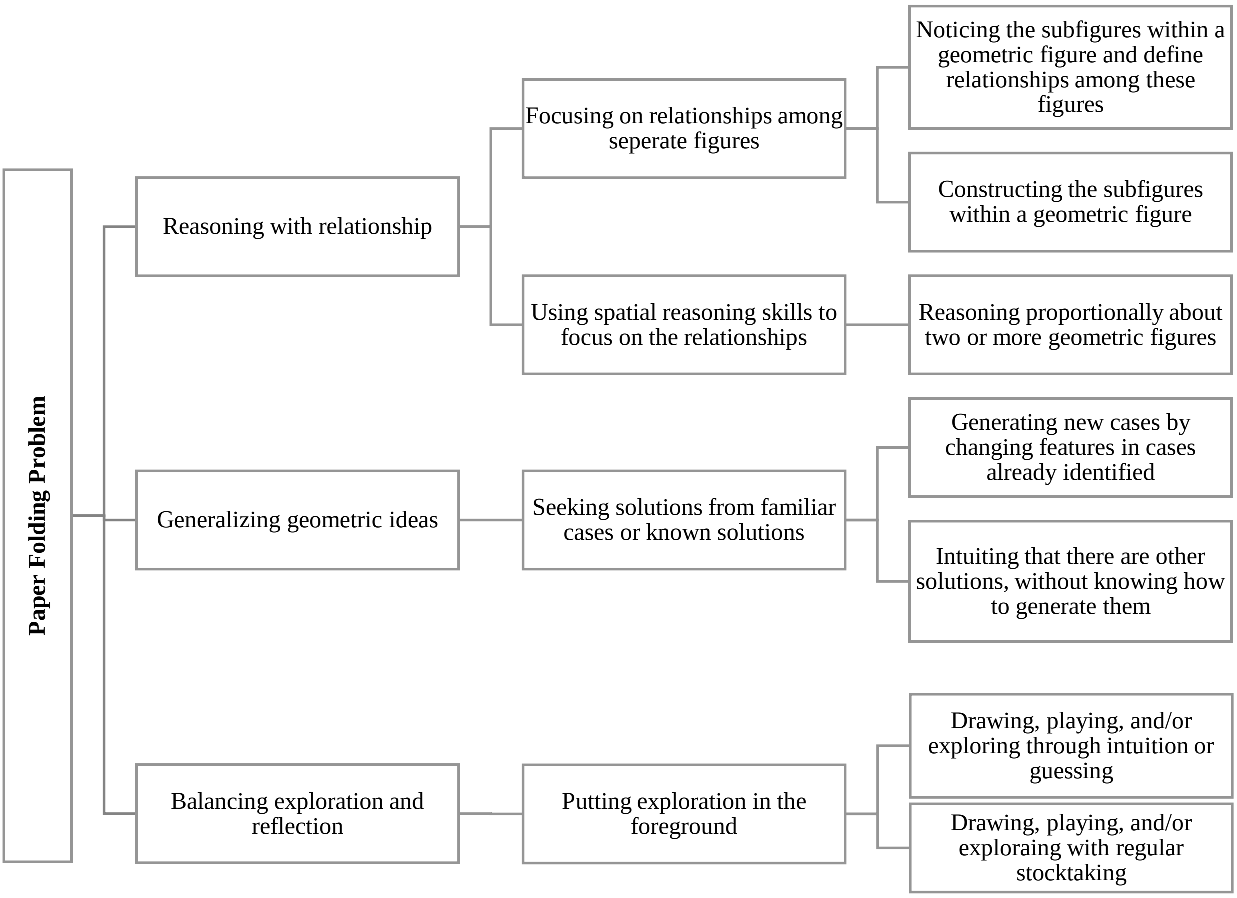


Figure 2: *Resulting Habits of Mind in Investigating Area by Paper Folding Problem.*

The students were given the polygons in Figure 3 for the second problem, "Finding Area with Different Ways," and were asked to calculate the areas of each polygon using the method of their choice, define at least three different ways to calculate the areas of polygon B, and explain whether or not the areas of other polygons could be calculated using these methods.

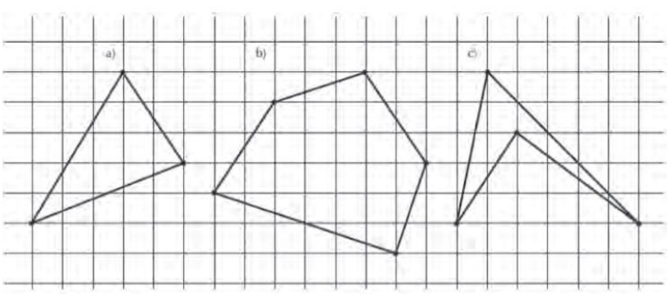
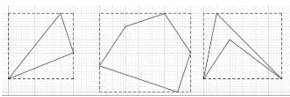
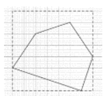
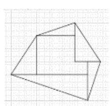
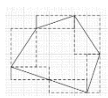
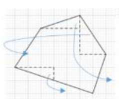
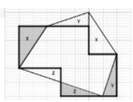
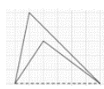


Figure 3: *The Polygons Given in Finding Area with Different Ways Problem*

Three students drew rectangles that encompass the polygons provided in the problem. They then calculated the solution by subtracting the areas of the polygonal regions outside the provided polygons from the area of the resulting rectangular regions. Thus, they realized the embedded geometric shapes within the rectangle and were able to calculate the desired area by creating connections among them. In this process, two of the students tried to calculate the areas by calculating the side lengths, trying to determine the heights, or forming right triangles. To calculate the area of polygon B with a different strategy, students divide the polygon into parts and collect the areas of the formed polygonal regions. Using another strategy, they drew rectangles where the oblique sides of the polygon were diagonals and added the quantity of perfect unit squares outside to them to half the areas of these rectangular regions. They also calculated the area of the figure by decomposing the shape into parts and swapping the parts that complement one another. In conclusion; while three of the students identified three strategies for calculating the area of polygon B, one was able to define two strategies. In the interview, it was observed that the students indicated that they could find the area of polygons A and C by drawing the rectangles encompassing the polygons, and that other methods were not suitable. In addition, to calculate the area of polygon C, two students instead created two nested triangles. They were able to get the desired result by subtracting the area of the small triangular region from the area of the large triangular region they had obtained. The solution approaches of the students and the resulting mental habits are shown in Table 4.

Table 4: *Students' Solution Approaches and Habits of Minds in Finding Area with Different Ways Problem*

Question	**Strategies**	**S1**	**S2**	**S3**	**S4**
By drawing the smallest rectangle enclosing the given polygons		-Reasoning with relationships A, B_1, C_1	-Reasoning with relationships A, B_1, C	-Reasoning with relationships A, B_2	-Reasoning with relationships A, C
By drawing a rectangle enclosing the given polygons					-Reasoning with relationships B_1
The sum of the areas of the triangular region formed by decomposing the polygon into parts.			-Reasoning with relationships B_2	-Reasoning with relationships B_1	-Reasoning with relationships B_2
By drawing a rectangle where the oblique sides of the polygon are diagonals		-Reasoning with relationships B_2		-Reasoning with relationships B_2	
By decomposing the polygon into polygonal parts and rearranging the parts	 	-Investigating invariants B_3			-Investigating invariants B_3
By creating two nested triangles		-Reasoning with relationships C_2		-Reasoning with relationships C	

Students, S2 and S4, used dynamic thinking differently from the other students and calculated the area of the figure by decomposing the shape into parts and rearranging the parts to complement each other. Additionally, S2 was unable to provide a different approach for the area of polygon B. Figure 4 shows, students' habits of minds and related indicators in the Finding Area with Different Ways problem.

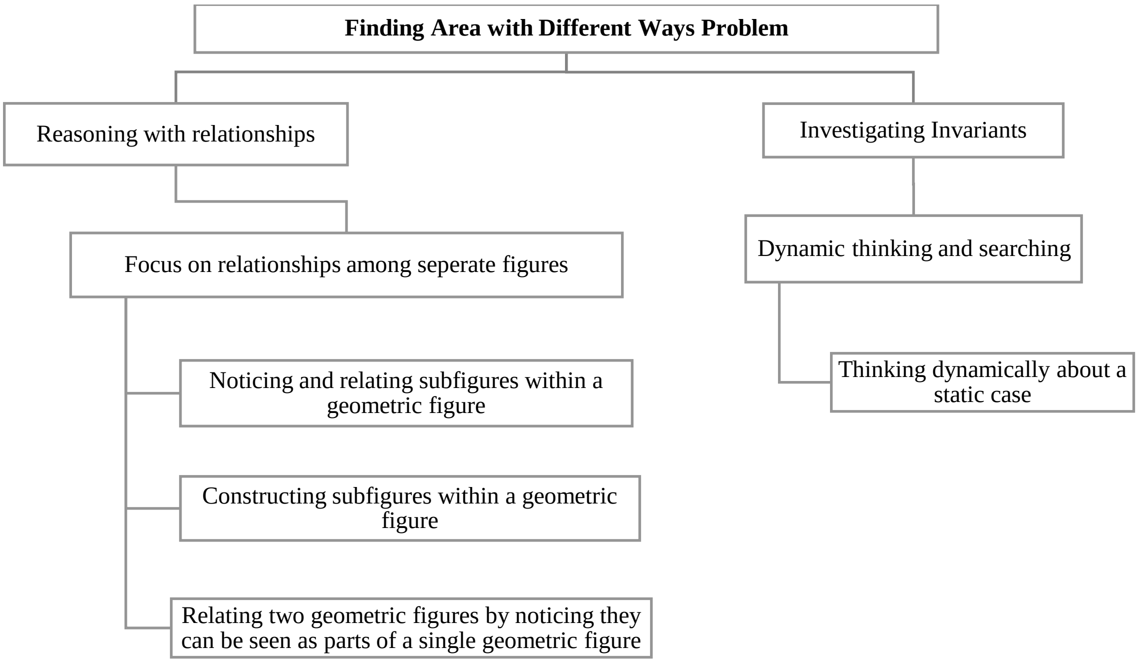


Figure 4. *Resulting Habits of Minds in Finding Area with Different Ways Problem*

For the area of the triangle problem, students asked additional problems of the finding the area with different ways problem (Table 1). They were asked to determine how to find all the coordinates for the third corner point, how many of the triangles formed are right and isosceles triangles, and the coordinates of the third corner point that forms these triangles. Finally, based on the responses, it was tried to determine whether they would change their initial response in any way and what they preferred to do in the event that they did change their initial response.

In the problem, it was intended for the students to recognize that the desired triangle's third vertex coordinates are on the $x=\pm 4$ lines starting from the area. They would also benefit from symmetry as they revealed this relationship. They were required to observe that the area did not change while the points on the line changed in the context of the investigating invariants habit. They were also required to calculate the coordinates of the vertices forming a total of four right triangles and ten isosceles triangles, taking into account previous instances of similar circumstances. The students' approaches and resulting habits of mind are shown in Table 5.

To determine the coordinates, the students first calculated the height of the triangle as 4 units, based on the area of the triangle. Two of the students first determined the coordinates of the third vertex as $(\pm 4, 4)$ and $(\pm 4, 10)$, using four right triangles with a right side length of 4 units. Additionally, every student stated that the I. or II. region should contain the third vertex of the triangle they discovered. They also agreed that after determining the points in either region, the symmetries would be taken to determine the points in the other region, and the points discovered would be located 4 units away from the y-axis.

One of the students did not specify any additional points past the four listed above until the study was over.

Table 5: *The Students' Solution Approaches (Ways) and Resulting Habits of Mind in Area of the Triangle Task*

Strategies	S1	S2	S3	S4
Determining the height as 4 units based on the area	-Reasoning with relationships	-Reasoning with relationships	-Reasoning with relationships	-Reasoning with relationships
Noticing the coordinate of the possible third vertices could be located on the lines x=4 and x=-4	-Reasoning with relationships -Generalizing geometric ideas -Investigating invariants -Balancing exploration and reflection	-Reasoning with relationships -Generalizing geometric ideas -Investigating invariants -Balancing exploration and reflection	-Reasoning with relationships -Generalizing geometric ideas -Investigating invariants -Balancing exploration and reflection	
Determining the third vertex as points on line segments with endpoints (±4,4) and (±4,10)				-Reasoning with relationships -Investigating invariants -Balancing exploration and reflection
Identifying right triangles with the area of 12 units square (a total of four)	-Reasoning with relationships -Generalizing geometric ideas -Investigating invariants	-Reasoning with relationships -Generalizing geometric ideas -Investigating invariants	-Reasoning with relationships -Generalizing geometric ideas -Investigating invariants	-Reasoning with relationships -Generalizing geometric ideas -Investigating invariants
Identifying isosceles triangles with an area of 12 units square (a total of 10)		-Reasoning with relationships -Generalizing geometric ideas -Investigating invariants -Balancing exploration and reflection		
Identifying some of the isosceles triangles with an area of 12 units square	-Reasoning with relationships -Investigating invariants -Balancing exploration and reflection (6 triangles)		-Reasoning with relationships -Investigating invariants -Balancing exploration and reflection (6 triangles)	-Reasoning with relationships -Investigating invariants -Balancing exploration and reflection (2 triangles)

All of the students were able to identify four right triangles with third vertices (±4, 4) and (±4, 10) and two isosceles triangles with vertex (±4, 7). Additionally, a student recognized that eight isosceles triangles with congruent lengths of 6 units were possible, but s/he could not determine the coordinates of the third vertex of these triangles and claimed that the ordinate was not an integer. Despite being able to recognize that there could be four acute-angled isosceles triangles with congruent lengths of 6 units, one of the students could not determine the coordinates of the third vertex of these triangles. Another student could see that there could be four isosceles triangles with congruent lengths of six units. However, he was unable to determine the coordinates of the third vertex of these triangles and claimed that the ordinate was not an integer. One student could not identify other than the two isosceles triangles mentioned above.

At the end of the study, three of the students stated that the desired points were on the x=±4 lines, while one of these students stated that the far points would not form a triangle due to the triangle inequality. A student, on the other hand, was unable to go any further and determined the solutions to be all points on the line segment with the end points of (4, 4) and (4, 10). The resulting habits of mind of the students and related indicators are shown in Figure 5.

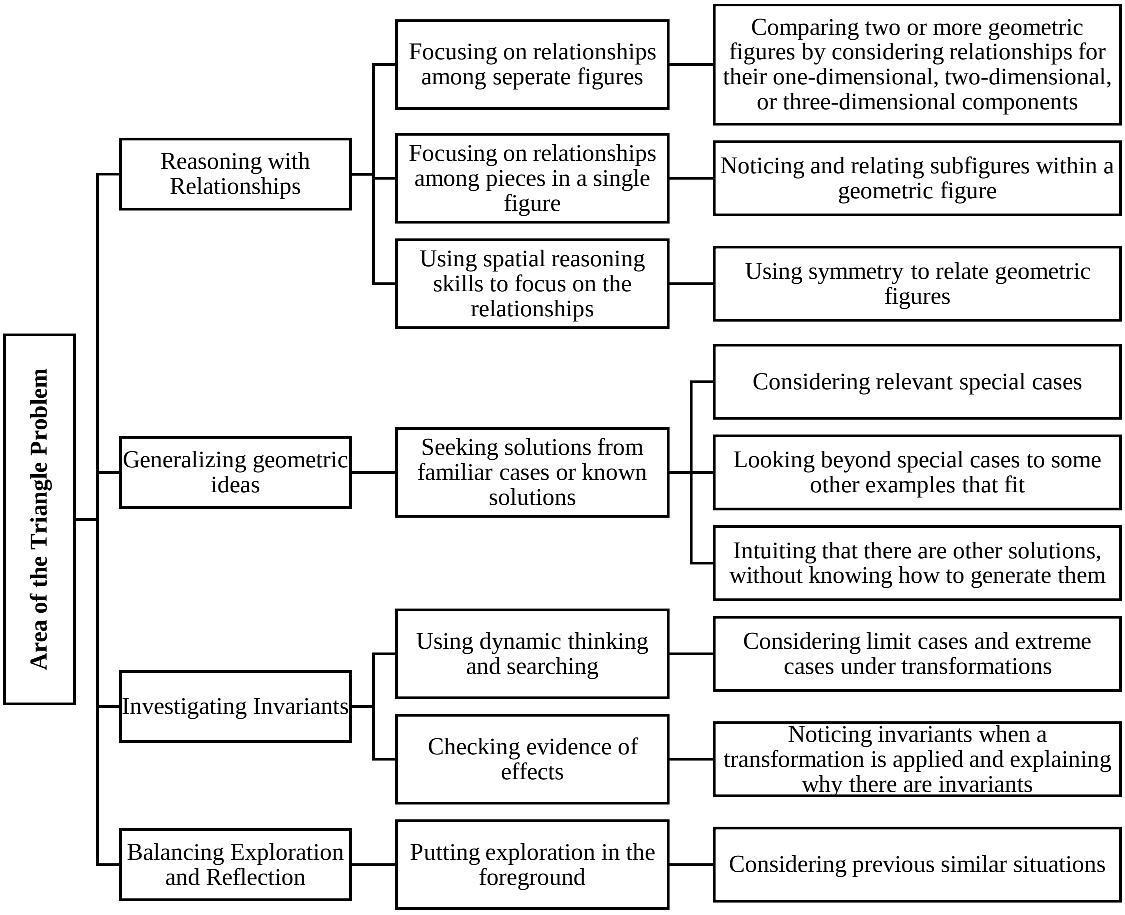


Figure 5. *Resulting Habits of Mind in Area of the Triangle Problem*

4. DISCUSSION and RESULTS

Considering the three problems, it can be seen that the students were able to create relationships between the parts of a shape and itself, and between the components of these shapes with various geometric sizes. Additionally, they were able to identify the connections between geometric shapes by using their unique reasoning skills (symmetry).

It can be said that the students are successful in terms of the generalization component in the questions that require different answers in the same context. The generalization skill, which is not easy for students, can be developed by developing different strategies, producing different solutions to the same problem, or recognizing that there might be other solutions (Driscoll, 2007). The results of the study showed that the questions that led to generalization were the ones on which the students spent the most time. One specific example of questions with this particular focus was in the investigating area by paper folding task.

Two students calculated the area using different ways by first figuring out where the right triangles' third vertex should be, and then they were able to enhance their answers. When the researcher provided the non-integer ordinates as an example, a student who had initially determined only 4 right triangles in the area calculation problem could foresee the next steps for his solution.

The students presented the indicators of the investigating invariants component in general. This habits of mind can develop when someone decides to experiment with a geometric transformation of the structures while taking into account both changing and static features (Driscoll, 2007). In the findings of the area with a different way problem, two students could think dynamically by decomposing the polygon into parts and rearranging the parts. All students, except one, were able to realize that the third vertex was on the $x = \pm 4$ line in the area calculation problem, but one of them reached a generalization that the far points would not satisfy the triangle inequality. The other students claimed that on the line segments with endpoints $(\pm 4, 4)$ and $(\pm 4, 10)$, the coordinate of the third vertex shifts. Although the students could determine the coordinates of the third vertex of every right triangle in this situation, they did so for two of the isosceles triangles. One of the students identified the other eight and the other one identified four isosceles triangles, but s/he could not identify the coordinates. The area calculation by decomposing the polygon into parts and moving the parts reveals the investigating invariants component when it is thought that the third vertex is above the $x = 4$ line.

A geometric problem must be repeatedly reviewed and assessed as part of the investigation and reflection component. This skill must be developed to a sufficient level in order for other components to develop (Köse ve Tanışlı, 2014). While engaging with investigating the area by paper folding task and area calculation problems during the clinical interviews, participants made helpful or unhelpful folds and drawings, checked their solution process by regularly assessing the situation, and fostered thinking by taking into account prior instances of situations that were similar.

In terms of geometric habits of mind, each participant can generally be said to be in a good place when compared to studies of a similar nature. It is believed that the differences between the findings of this study, and Driscoll et al. (2007) and Köse and Tanışlı (2014) are the result of the participants' selection through purposeful sampling.

Declaration of Contribution Rate of Authors

The authors contributed equally to the research.

Conflict of Interest Declaration

The authors declare no potential conflicts of interest with respect to the research, authorship and/or publication of this article.

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