ANADOLU ÜNİVERSİTESİ

Bilim ve Teknoloji Dergisi B-Teorik Bilimler
Cilt: 2 Sayı: 2 2013
Sayfa:111-120

ARAŞTIRMA MAKALESİ / **RESEARCH ARTICLE**

Fatih KIZILASLAN[1], Mustafa NADAR[2]

CLASSICAL AND BAYESIAN ANALYSIS FOR THE GENERALIZED EXPONENTIAL DISTRIBUTION BASED ON RECORD VALUES AND TIMES

ABSTRACT

In this study, the maximum likelihood and Bayes estimators for the shape parameter of two-parameter generalized exponential distribution are obtained based on the record values with the number of trials following the record values (inter-record times) when the scale parameter is known. By using a Monte Carlo simulation methods: (i) the maximum likelihood and Bayes estimators are compared in terms of the estimated risk (ii) the estimators are compared with and without the inter-record times are taken into consideration. Moreover, a real life data is used to illustrate the results which were derived.

Keywords: Generalized exponential distribution, Record values and times, Bayesian estimation, Symmetric and asymmetric loss functions

REKOR DEĞERLER VE ZAMANLARINA GÖRE GENELLEŞTİRİLMİŞ ÜSTEL DAĞILIM İÇİN KLASİK VE BAYESÇİ ANALİZ

ÖZ

Bu çalışmada, iki parametreli genelleştirilmiş üstel dağılımın biçim parametresi için en çok olabilirlik ve Bayes tahmin edicileri ölçek parametresi bilindiğinde rekor değerler ve deneme zamanları göz önünde bulundurularak elde edilmiştir. Monte Carlo simülasyonu kullanılarak: (i) en çok olabilirlik ve Bayes tahmin edicileri ortalama karesel hatalarına göre karşılaştırıldı, (ii) rekor değerlerin deneme zamanları dikkate alınarak ve alınmaksızın tahmin ediciler karşılaştırıldı. Ayrıca, elde edilen sonuçları örnekle açıklamak için gerçek bir veri seti kullanıldı.

Anahtar Kelimeler: Genelleştirilmiş üstel dağılım, Rekor değerler ve zamanları, Bayesçi yaklaşım, Simetrik ve asimetrik kayıp fonksiyonları

[1,] Gebze Institute of Technology, Department of Mathematics, 41400, Çayırova, Kocaeli, Turkey.
Tel: 0262 605 13 72, E-mail: kizilaslan@gyte.edu.tr

[2,] Gebze Institute of Technology, Department of Mathematics, 41400, Çayırova, Kocaeli, Turkey.
Tel: 0262 605 13 68, E-mail: nadar@gyte.edu.tr

Recieved: 28 November 2012 **Revised:** 19 April 2013 **Accepted:** 18 June 2013

1. INTRODUCTION

The two-parameter generalized exponential (GE) distribution was introduced by Gupta and Kundu (1999). They studied various properties of the model and observed those many of the properties are quite similar to those of the gamma family and the Weibull family. If a random variable X is distributed two-parameter GE, denoted by $GE(\alpha,\lambda)$, then its cumulative distribution function (cdf) is

$$F(x;\alpha,\lambda) = (1-e^{-\lambda x})^{\alpha},\ x>0, \tag{1}$$

with the corresponding probability distribution function (pdf) as

$$f(x;\alpha,\lambda) = \alpha\lambda e^{-\lambda x}(1-e^{-\lambda x})^{\alpha-1},\ x>0, \tag{2}$$

where $\alpha>0$ and $\lambda>0$ are the shape and scale parameters, respectively. There are vast literatures when the underlying distribution is GE. For example, the single and product moments of order statistics and the best linear unbiased estimators of the location and scale parameters with known shape parameter were derived by Raqab and Ahsanullah (2001). The exact expressions for single and product moments of record statistics and the best linear unbiased estimators of the location and scale parameters were also obtained by Raqab (2002). The Bayes and empirical Bayes estimators for the shape parameter based on record values and prediction bounds for future lower record values were obtained by Jaheen (2004). The estimation of stress-strength reliability was derived by Kundu and Gupta (2005) when the underlying distributions are independent and have the same scale parameters. The Bayesian estimation and prediction of the parameters based on complete and type II censored samples were considered by Raqab and Madi (2005). The Bayesian estimation and prediction of the parameters based on lower record values were discussed by Madi and Raqab (2007). The Bayes estimates of the parameters were obtained by Kundu and Gupta (2008). The minimum variance unbiased estimator, the maximum likelihood estimator and the Bayesian estimator for the shape parameter based on k-th lower record values were obtained by Malinowska and Szynal (2009).

Let $X_1, X_2, \ldots$ be a sequence of continuous random variables. An observation X_j is called an upper record value if it exceeds that of than all previous observation. By definition, X_1 is an upper record value. We can give an analogous definition for the lower record values. A record data may be represented by $(\underline{R},\underline{K}) := (R_1, K_1, R_2, K_2, \ldots, R_m, K_m)$, where R_i is the ith record value, meaning new maximum (or minimum), and K_i is the number trials following the observation of R_i that are needed to obtain a new record value R_{i+1}, which is called inter-record times. There are two sampling schemes for generating such a record-breaking data known as inverse sampling scheme and random sampling scheme. Under the inverse sampling scheme, units are taken sequentially and sampling is terminated when the mth maximum is observed. In this case, the total number of units sampled is a random number and K_m is defined to be one for convenience. Under the random sampling scheme, a random sample $X_1, X_2, \ldots, X_n$ is examined sequentially and successive maximum values are recorded. In this setting, we have $N^{(n)}$, the number of records obtained, to be random and given a value of m, $\sum_{i=1}^{m} K_i = n$.

In recent years there has been a growing interest in the study of inference problems associated with record values. For example, the Bayesian estimation for the two parameters of some life distributions, including exponential, Weibull, Pareto and Burr Type XII, based on upper record values were considered by Ahmadi and Doostparast (2006).

The Bayes estimates for the two parameters of Burr Type XII distribution on the basis of a linear exponential loss function were derived by Nadar and Papadopoulos (2011). Statistical analysis of record values from the Kumaraswamy distribution was considered by Nadar et al. (2013).

When the underlying distribution is exponential, estimation of the mean parameter was obtained by Sameniego and Whitaker (1986) under random sampling scheme and inverse sampling scheme. Non-Bayesian and Bayesian estimates were derived for the two parameters of the exponential distribution based on record values and their corresponding inter-record times under the inverse sampling scheme by Doostparast (2009). The optimal random sampling plan and associated cost analysis for exponential distribution were studied by Doostparast and Balakrishnan (2010). When the underlying distribution is lognormal, non-Bayesian and Bayesian estimates of the parameters were obtained by Doostparast et. al. (2012).

In the literature, the maximum likelihood and Bayes estimates for one and two parameters GE distribution based on the lower record values were derived by Jaheen (2004) and Madi and Raqab (2007), respectively. In this paper, we obtained the shape parameter estimations for the GE distribution using upper record values with their corresponding inter-record times under the classical and Bayesian frameworks when the scale parameter is known. For the sake of comparison we also obtain the estimates based on the upper record values without considering inter-record times. Finally, Monte Carlo simulations are performed to observe the effect of the inter-record times in estimations.

The paper is organized as follows. In Section 2, we derive the maximum likelihood estimation (MLE) of the parameters under the inverse sampling scheme. In Section 3, when the scale parameter λ is known, we obtain the Bayesian estimations of the shape parameter α under the symmetric and asymmetric loss functions. In Section 4, a computer simulation study is done to compare the different estimators discussed in early sections and the results are reported. Moreover, real data is used to illustrate the findings. Finally, we conclude the paper in Section 5.

2. CLASSICAL ESTIMATION

In this section, we consider the parameter estimation of GE distribution under inverse sampling scheme.

2.1. Inverse Sampling Scheme

Let $X_1, X_2, ...$ be independent identical distributed (i.i.d.) random variables, each drawn from a population with cdf $F(.)$ and pdf $f(.)$. Then the likelihood function associated with the sequence $(R_1, K_1, R_2, K_2, \ldots, R_m, K_m)$ is given by Hofmann and Nagaraja (2003)

$$L(r,k) = \prod_{i=1}^{m} f(r_i)\{F(r_i)\}^{k_i-1} I_{(r_{i-1},\infty)}(r_i) \tag{3}$$

where $r_0 \equiv -\infty$, $k_m \equiv 1$ and $I_A(x)$ is the indicator function of the set A. From the equations (1)- (3), we have

$$L(\alpha,\lambda;r,k) = \alpha^m \lambda^m \exp\left\{-\lambda\sum_{i=1}^{m} r_i + \alpha\sum_{1=1}^{m} k_i \ln(1-e^{-\lambda r_i}) - \sum_{i=1}^{m}\ln(1-e^{-\lambda r_i})\right\}, \ -\infty < r_1 < \ldots < r_m \tag{4}$$

and so the log-likelihood function is

$$l(\alpha,\lambda;r,k) = m\ln\alpha + m\ln\lambda - \lambda\sum_{i=1}^{m} r_i + \alpha\sum_{1=1}^{m} k_i \ln(1-e^{-\lambda r_i}) - \sum_{i=1}^{m}\ln(1-e^{-\lambda r_i}). \tag{5}$$

The maximum likelihood estimates (MLEs) of α and λ are given by

$$\hat{\alpha} = -\frac{m}{\sum\limits_{1=1} K_i \ln(1\text{-}\mathrm{e}^{-\hat{\lambda}R_i})} \tag{6}$$

and $\hat{\lambda}$ is the solution of the following non-linear equation

$$\frac{m}{\lambda} - \sum_{i=1}^{m} r_i + \sum_{1=1}^{m} \frac{r_i e^{-\lambda r_i}}{\ln(1-e^{-\lambda r_i})}(\hat{\alpha} k_i - 1) = 0.$$

Therefore, $\hat{\lambda}$ can be obtained as the solution of the non-linear equation of the form $h(\lambda) = \lambda$, where

$$h(\lambda) = m\left[\sum_{1=1}^{m} \frac{r_i}{1-e^{-\lambda r_i}} + \sum_{i=1}^{m} \frac{k_i r_i e^{-\lambda r_i}}{1-e^{-\lambda r_i}} \frac{m}{\sum\limits_{1=1}^{m} k_i \ln(1-e^{-\lambda r_i})}\right]^{-1}. \tag{7}$$

Since $\hat{\lambda}$ is a fixed point solution of the non-linear equation (7), its value can be obtained using an iterative scheme as like:

$$h(\lambda_{(j)}) = \lambda_{(j+1)}, \tag{8}$$

where $\lambda_{(j)}$ is the j th iterate of $\hat{\lambda}$. The iteration procedure should be stopped when $\left|\lambda_{(j)} - \lambda_{(j+1)}\right|$ is sufficiently small.

2.2. MLE Estimation When λ is Known

In this case, we assume that $\lambda = 1$ without loss of generality. Then, we have from (4)

$$L(\alpha,1;r,k) = \alpha^m \exp\left\{-\sum_{i=1}^{m} r_i + \alpha \sum_{1=1}^{m} k_i \ln(1-e^{-r_i}) - \sum_{i=1}^{m} \ln(1-e^{-r_i})\right\}, \; -\infty < r_1 < \ldots < r_m. \tag{9}$$

It is clear that U is a complete sufficient statistic for α and the MLE of α is $\hat{\alpha}_M = \frac{m}{U}$ where $U = -\sum\limits_{1=1}^{m} K_i \ln(1-e^{-R_i})$. The distribution of $\hat{\alpha}_M$ can be obtained by using the moment generating function of U, which is given as $M(t) = \frac{1}{\left(1-\frac{t}{\alpha}\right)^m}$, $\alpha > t$. Therefore, U is distributed Gamma with parameters (m,α) with the pdf

$$f(u) = \frac{\alpha^m}{\Gamma(m)} u^{m-1} e^{-\alpha u}, \; u > 0.$$

It is easily seen that $E(\hat{\alpha}_M) = \dfrac{m\alpha}{m-1}$ and an unbiased estimator of α is given by $\hat{\alpha}_U = \dfrac{m-1}{U}$ Moreover, $\hat{\alpha}_U$ is a best unbiased estimator from Lehmann-Scheffé Theorem.

3. BAYESIAN ESTIMATION

Bayesian approach has a number of advantages over the conventional frequentist approach. Bayes theorem is the only consistent way to modify our beliefs about the parameters given the data that actually occurred. The beliefs about the parameter are called prior distribution. Any prior information about the parameters is considerably useful. We need to some prior distributions of the unknown parameters for the Bayesian inference. In this section, we consider the Bayes estimate of the shape parameter α when the scale parameter λ is known.

We assume that α has a Gamma prior with parameters (a_1, b_1) and its pdf denoted by $\pi(\alpha)$. Then, the posterior density function of α is

$$\pi(\alpha | r, k) = \frac{L(\alpha; r, k)\pi(\alpha)}{\int_0^\infty L(\alpha; r, k)\pi(\alpha) d\alpha} = \frac{(b_1 + U)^{m+a_1}}{\Gamma(m + a_1)} \alpha^{m+a_1-1} e^{-\alpha(b_1+U)},$$

that is $\alpha | r, k$ is distributed Gamma with parameters $(m + a_1, b_1 + U)$. We know that the Bayes estimate of α under squared error (SE) loss function, $\hat{\alpha}_{BS}$, is the mean of the $\alpha | r, k$. Therefore,

$$\hat{\alpha}_{BS} = \frac{m + a_1}{b_1 + U}. \tag{10}$$

It is well known that the use of symmetric loss functions may be inappropriate in many circumstances, particularly when positive and negative errors have different consequences. The use of asymmetrical loss function, which associates greater importance to overestimation or underestimation, can be considered for the estimation of the parameter. A number of asymmetric loss functions are proposed for use, among these, one of the most popular asymmetric loss functions is linear-exponential loss function (LINEX), was introduced by Varian (1975). The LINEX loss function can be expressed as

$$L(\theta, \delta) = e^{v(\delta - \theta)} - v(\delta - \theta) - 1, \ v \neq 0,$$

where δ is an estimator of θ. The sign and magnitude of v represents the direction and degree of asymmetry, respectively. If $v > 0$, the overestimation is more serious than underestimation, and vice versa. For v close to zero, the LINEX loss is approximately SE loss and almost symmetric. It is easily seen that the value of $\delta(X)$ that minimizes $E_{\theta|X}\left[L(\theta, \delta(X))\right]$ is

$$\hat{\delta}_{BL} = -\frac{1}{v}\ln\left(E_{\theta X}(e^{-v\theta})\right),$$

provided $E_{\theta|X}(e^{-v\theta})$ exists and is finite. Here $E_{\theta|X}(.)$ denotes the expected value with respect to the posterior density function θ given X .

Therefore, the Bayes estimate of α under the LINEX loss function, $\hat{\alpha}_{BL}$, for our case is given by

$$\hat{\alpha}_{BL} = -\frac{1}{v}\ln E_{\alpha|(r,k)}(e^{-v\alpha}) = -\frac{1}{v}\ln\int_0^{\infty} e^{-v\alpha}\pi(\alpha \mid r,k)d\alpha = \frac{m+a_1}{v}\ln\left(1+\frac{v}{b_1+U}\right). \tag{11}$$

If we use the Jeffrey's non-informative prior, that is $\pi(\alpha) = 1/\alpha$, then we have that $\alpha|r,k$ is distributed Gamma with parameters (m,U). Hence, the Bayes estimates of α under the SE and LINEX loss functions are obtained as

$$\hat{\alpha}_{BS,0} = \frac{m}{U},\ \hat{\alpha}_{BL,0} = \frac{m}{v}\ln\left(1+\frac{v}{U}\right). \tag{12}$$

4. SIMULATION STUDY

In this section, we present the analysis of two data sets. The first data set is artificial and the second is a real life one.

4.1 Simulated Data

In order to compare the different estimators, Monte Carlo simulations are performed by using different sample sizes and different priors. All the programs are written in Matlab 2010a. All the results are based on 1000 replications. The estimated risk (ER) of θ, when $\hat{\theta}$ is estimated by θ, is given by $ER(\theta) = \frac{1}{N}\sum_{i=1}^{N}(\hat{\theta}_i - \theta_i)^2$ under the SE loss function. Moreover, the estimated risk of θ under the LINEX loss function is given by. $ER(\theta) = \frac{1}{N}\sum_{i=1}^{N}\left(e^{v(\hat{\theta}_i - \vartheta_i)} - v(\hat{\theta}_i - \theta_i) - 1\right)$.

In the Table 1, we consider the case where $\lambda = 1$ and α has Gamma prior with parameters $(a_1, b_1) = (10,5)$ and $(a_1, b_1) = (16,6)$. When the estimates obtained without taking inter-record times into consideration, the results are given in Table 2 and is denoted by α^*. The ML and Bayesian estimates for SE and LINEX loss functions are listed in both Tables 1 and 2.

In Tables 1 and 2, it is observed that as the sample size increases the estimated risk of the estimates generally decrease. The ERs of the MLEs are greatest among all estimators. Moreover, the ERs of the Bayes estimators under the SE loss function are smaller than the MLEs, as expected. Furthermore, it is observed that the ERs for estimates of α are smaller than that of α^*. It is quite natural to see such a result when more information is available. As a result, the simulation illustrates that considering inter-record times is increasing the accuracy and the precision of the estimates.

Table 1. Estimations of α and ERs when $\lambda = 1$ by upper record values with considering inter-record times.

(a_1, b_1)	m	α	$\hat{\alpha}_M$	$\hat{\alpha}_U$	$\hat{\alpha}_{BS}$	$\hat{\alpha}_{BL}$				
						v	-2	-1	1	2
(10,5)	3	1.9931	3.0195	2.0130	2.0020		2.4069	2.1788	1.8595	1.7413
			8.0356	3.1027	0.3329		0.8988	0.1918	0.1468	0.5235
	5		2.5858	2.0686	2.0096		2.3552	2.1633	1.8826	1.7751
			3.1378	1.7632	0.2881		0.7410	0.1626	0.1288	0.4627
	8		2.2811	1.9960	1.9992		2.2762	2.1251	1.8920	1.7991
			0.9970	0.6900	0.2341		0.5551	0.1266	0.1088	0.4052
	10		2.1585	1.9427	1.9752		2.2148	2.0853	1.8798	1.7961
			0.6242	0.4840	0.2341		0.5551	0.1266	0.1088	0.4052
(16,6)	3	2.6295	4.1175	2.7450	2.6621		3.1303	2.8704	2.4900	2.3447
			43.0376	18.0657	0.4004		0.9879	0.2212	0.1840	0.6852
	5		3.2998	2.6398	2.6457		3.0575	2.8312	2.4898	2.3562
			5.4854	3.2035	0.3545		0.9297	0.1967	0.1632	0.6062
	8		3.0307	2.6519	2.6505		3.0054	2.8126	2.5115	2.3905
			2.0975	1.4729	0.3005		0.7477	0.1653	0.1395	0.5239
	10		2.8375	2.5538	2.6272		2.9449	2.7735	2.5001	2.3883
			1.0057	0.7888	0.2870		0.7147	0.1581	0.1337	0.5062

The first and second rows represent the average estimates and estimated risks.

Table 2. Estimations of α and ERs when $\lambda = 1$ by upper record values without inter-record times.

(a_1, b_1)	m	α	α_M^*	α_{BS}^*	α_{BL}^*				
					v	-2	-1	1	2
(10,5)	3	1.9931	3.9281	2.0868		2.1171	2.1645	1.4116	1.4591
			36.1005	0.3712		1.4297	0.2126	0.2374	0.6736
	5		3.8867	2.0940		2.1248	2.1722	1.4165	1.4639
			27.5305	0.3704		1.3789	0.2108	0.2362	0.6739
	8		3.7863	2.0899		2.1205	2.1679	1.4138	1.4612
			24.7304	0.3716		1.4819	0.2165	0.2364	0.6720
	10		3.5952	2.0702		2.0995	2.1469	1.4009	1.4483
			19.2838	0.3547		1.4428	0.2072	0.2380	0.6711
(16,6)	3	2.6295	4.9898	2.8882		2.8628	2.9469	2.0031	2.0872
			79.2675	0.4908		1.5302	0.2443	0.2742	0.7696
	5		5.1286	2.8831		2.8576	2.9416	1.9996	2.0837
			59.4947	0.4906		1.5184	0.2436	0.2756	0.7717
	8		5.1273	2.8875		2.8621	2.9462	2.0027	2.0868
			50.1313	0.4862		1.5559	0.2425	0.2733	0.7662
	10		4.6478	2.8857		2.8603	2.9443	2.0015	2.0855
			35.9188	0.4888		1.7374	0.2471	0.2738	0.7623

The first and second rows represent the average estimates and estimated risks.

4.2 Real Life Data

In this example we present a data analysis of the amount of annual rainfall (in inches) during Januuary recorded at Los Angeles Civic Center from 1900 to 2006 (see the website of Los Angeles Almanac: www.laalmanac.com/weather/we08aa.htm). The upper record values of this data and their corresponding inter-record times are given in Table 3.

Table 3. Record and inter-record times data arising from annual rainfall data $m = 8$

i	1	2	3	4	5	6	7	8
R_i	2.49	2.57	3.85	7.02	7.27	10.35	13.3	14.94
K_i	4	1	1	2	5	2	53	1

The sample is tested if the underlying distribution is $GE(\hat{\alpha}, 0.25)$ by using the Kolmogorov-Smirnov (K-S) test. We compute the K-S distance between the empirical distribution and the fitted distribution functions when the parameters are obtained by MLE and Bayesian methods. All results about this data are presented in Table 4. It is observed that the GE distribution when $\lambda = 0.25$ provides an adequate fit for data. The goodness of fit test strongly suggest to use the Bayesian estimates for the parameter α, because of bigger p values.

Table 4. The estimates of α, Kolmogorov-Smirnov distances and the corresponding p-values between, the empirical and the fitted distribution function when $\lambda = 0.25,\ a_1 = 2.9879,\ b_1 = 0.3868$.

Methods	Estimates	K-S distance	p-value
MLE	1.0400	0.4494	$p \approx 0.05$
Bayes (SEL)	1.3600	0.3975	$0.1 < p < 0.2$
Bayes(LINEX,v=-2)	1.5626	0.3683	$0.1 < p < 0.2$
Bayes(LINEX,v=-1)	1.4518	0.3841	$0.1 < p < 0.2$
Bayes(LINEX,v=1)	1.2822	0.4089	$0.1 < p < 0.2$
Bayes(LINEX,v=2)	1.2151	0.4190	$0.05 < p < 0.1$

5. CONCLUSION

In this study, we compared the different estimations for the shape parameter when the scale parameter is known for the two-parameter generalized exponential distribution. It is observed that the Bayesian estimators have a smaller estimated risk and this result does not change for the different values of the prior parameters in Monte Carlo simulation. Moreover, the simulation illustrates why the inter-record times should be considered. For the real life data prior parameters can be choosen by method of moments. An application of goodness of fit test on annual rainfall data suggests to use Bayesian estimations of the parameter α.

REFERENCES

Ahmadi, J. and Doostparast, M. (2006). Bayesian Estimation and Prediction for some Life Distributions Based on Record Values. *Statistical Papers* 47 (3): 373–392.

Doostparast, M. (2009). A Note on Estimation Based on Record Data. *Metrika*, 69: 69-80.

Doostparast, M. and Balakrishnan, N. (2010). Optimal Sample Size for Record Data and Associated Cost Analysis for Exponential Distribution. *Journal of Statistical Computation and Simulation*, 80 (12): 1389-1401.

Doostparast, M., Deepak, S. and Zangoie, A. (2012). Estimation with The Lognormal Distribution on The Basis of Records. *Journal of Statistical Computation and Simulation*, DOI:10.1080/00949655.2012.691973.

Gupta, R. D. and Kundu, D. (1999). Generalized Exponential Distributions. Austral. New Zealand *J.Statistics*, 41: 173-188.

Hofmann, G. and Nagaraja, H.N. (2003). Fisher Information in Record Data. *Metrika*, 57: 177-193.

Jaheen, Z.F. (2004). Empirical Bayes Inference for Generalized Exponential Distribution Based on Records. *Communications in Statistics-Theory and Methods*, 33(8): 1851–1861.

Kundu, D. and Gupta, R.D. (2005). Estimation of P(Y<X) for Generalized Exponential Distribution. *Metrika*, 61: 291-308.

Kundu, D. and Gupta, R.D. (2008). Generalized Exponential Distribution: Bayesian Estimations. *Computational Statistics and Data Analysis*, 52: 1873-1883.

Madi, M.T. and Raqab, M.Z. (2007). Bayesian Prediction of Rainfall Records using The Generalized Exponential Distribution. *Environmetrics*, 18: 541-549.

Malinowska, I. and Szynal, D. (2009). Inference and Prediction for A Generalized Exponential Distribution Based on The k-Th Lower Records. *International Journal of Pure and Applied Mathematics*, 52(2): 211-227.

Nadar, M. and Papadopoulos, A. (2011). Bayesian Analysis for The Burr Type XII Distribution Based on Record Values. *Statistica*, 71(4): 421-435.

Nadar, M., Papadopoulos, A. and Kızılaslan, F. (2013). Statistical Analysis for Kumaraswamy's Distribution Based on Record Data. *Statistical Papers*, 54: 355-369.

Raqab, M. Z., Ahsanullah, M. (2001). Estimation of The Location and Scale Parameters of Generalized Exponential Distribution Based on Record Statistics. *Journal of Statistical Computation and Simulation*, 69(2),109–124.

Raqab, M. Z. (2002). Inference for Generalized Exponential Distribution Based on Record Statistics. *Journal of Statistical Planning and Inference*, 104, 339-350.

Raqab, M.Z. and Madi, M.T. (2005). Bayesian Inference for The Generalized Exponential Distribution. *Journal of Statistical Computation and Simulation*, 75 (10): 841-852.

Samaniego, F.J. and Whitaker, L.R. (1986). On Estimating Population Characteristics from Record-Breaking Observations. I. Parametric Results. *Naval Research Logistics Quarterly*, Vol. 33, 531-543.

Varian, H. R. (1975). *A Bayesian Approach to Real Estate Assessment.* In: Finberg SE, Zellner A.(eds) Studies in Bayesian Econometrics and Statistics in Honor of Leonard J. Savege. North Holland, Amesterdam : 195.208.