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DETERMINATION OF EDGES OF A CONVEX POLYTOPE

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ABSTRACT

In this paper, the problem of determination of the edges of a convex polytope is considered. It is shown that this problem is equivalent to the standard linear programming problem and therefore can be solved by the simplex method. Further, for a special type of polytopes which are an affine transformation of a box we show that extremal points determine edges.

Keywords: Polytopes, Extreme point, Edge, Edge Theorem.

BİR KONVEKS POLİTOPUN KENARLARININ BELİRLENMESİ

ÖZ

Bu çalışmada, konveks bir politopun kenarlarının belirlenmesi problemi ele alınmıştır. Bu problemin, simpleks yöntemiyle çözülebilen bir standart lineer programlama problemine denk olduğu gösterilmiştir. Ayrıca, bir kutunun afin dönüşüm altındaki görüntüsü olan özel politoplar için uç noktaların, politopun kenarlarını belirlediği gösterilmiştir.

Anahtar Kelimeler: Politoplar, Uç nokta, Kenar, Kenar Teoremi.

1. INTRODUCTION

The frame problem, i.e. the problem of determination of extremal points of a polytope has been studied in many works (see, for example Dulá et al. (1992), Dulá and Helgason (1996), Murty (2009), Rosen et al. (1992), Wallace and Wets (1967), Ziegler (2006)), while the problem of determination of edges has been paid little attention.

The problem of determination of edges of a polytope has mathematical interest and also is important for applications. For example, consider the robust stability problem for a polynomial polytope. Since every polynomial can be interpreted as coefficient vector, polynomial polytope can be interpreted as a polytope in a finite-dimensional space. Fundamental result in this theory is the Edge Theorem, which states that the polynomial polytope is (robustly) stable if all edges of the polytope are stable (Barmish (1994), Bartlett et al. (1988), Bhattacharyya et al. (1995)). Therefore, the recognition of the edges is important for this theory (Bhattacharyya et al. (1995), Pujara and Bollepalli (1994)).

In this work we consider the problem of determination of all edges of a convex polytope. In Section 2, we consider the general polytope, which is defined as convex hull of a finite number of points. We show that the line segment connecting two such points is an edge if and only if the optimal value of the corresponding linear programming problem is equal to 1 (Theorem 1).

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In Section 3, we consider the special types of polytopes which are important for applications. These polytopes are defined as an affine transformation of a box and called zonotopes. Such polytopes arise, for example, in the robust stability theory of an affine polynomial family, when every coefficient of the polynomial is an affine function of uncertainty parameters $q_1,\ q_2,\dots,q_n$ and the uncertainty vector $\mathbf{q}=(q_1,q_2,\dots,q_n)^T$ varies in a box Q (affine uncertainty structure). Here we show that the line segment joining the vertex points that are the images of an adjacent vertexes of the box Q is an edge (Theorem 4). Thus in such polytopes extremal points determine edges.

Now we recall some definitions from the convex analysis (Grünbaum (2003), Rockafellar (1970), Stoer and Witzgall (1970)). For the different elements $\mathbf{x},\mathbf{y}\in\mathbb{R}^n$ the line segment $[\mathbf{x},\mathbf{y}]$ is the set $\{(1-t)\mathbf{x}+t\mathbf{y} : t\in[0,1]\}$, and the straight line through $\mathbf{x}$ and $\mathbf{y}$ is the set $\{(1-t)\mathbf{x}+t\mathbf{y} : t\in\mathbb{R}\}$. In these definitions the vector $\mathbf{y}-\mathbf{x}$ is called the direction vector.

Let the finite set of different points $\mathcal{E}=\{\mathbf{a}_1,\mathbf{a}_2,\dots,\mathbf{a}_k\}$ be given. Then the convex hull of $\mathcal{E}$, i.e. the set

$$\mathcal{P}=\operatorname{conv}\mathcal{E}$$

is called a polytope and the set $\mathcal{E}$ is called the generator set of the polytope $\mathcal{P}$.

A point $\mathbf{x}\in\mathcal{P}$ is called an extremal point (vertex) of $\mathcal{P}$ if there does not exist $\mathbf{p}_1,\mathbf{p}_2\in\mathcal{P}$ with $\mathbf{p}_1\neq\mathbf{p}_2$ and $\lambda\in(0,1)$ such that $\mathbf{x}=(1-\lambda)\,\mathbf{p}_1+\lambda\,\mathbf{p}_2$.

Every extremal point of $\mathcal{P}$ is contained in $\mathcal{E}$ but non-every $\mathbf{a}_i$ from the set $\mathcal{E}$ is an extreme points of $\mathcal{P}$. The set of all $\mathbf{a}_i$ which are the extreme points of $\mathcal{P}$ is called the minimal generating set (the frame) of $\mathcal{P}$. A polytope is equal to the convex combination of its extreme points.

Let the polytope $\mathcal{P}$ be given. As usual, the edges of $\mathcal{P}$ are its 1-dimensional faces. We will use the following equivalent definition of edges (see Barmish (1994) p. 133, Papadimitriou and Steiglitz (1998) p. 61). Let $\mathbf{e}_1$ and $\mathbf{e}_2$ ($\mathbf{e}_1\neq\mathbf{e}_2$) be extreme points of $\mathcal{P}$. The segment $E=[\mathbf{e}_1,\mathbf{e}_2]$ is called an edge of $\mathcal{P}$ if the following condition holds: Given any $\mathbf{p}_1,\mathbf{p}_2\in\mathcal{P}$ with $\mathbf{p}_1,\mathbf{p}_2\notin[\mathbf{e}_1,\mathbf{e}_2]$ it follows that $[\mathbf{p}_1,\mathbf{p}_2]\cap[\mathbf{e}_1,\mathbf{e}_2]=\phi$.

For every edge E of the polytope $\mathcal{P}=\operatorname{conv}\{\mathbf{a}_1,\mathbf{a}_2,\dots,\mathbf{a}_k\}$ there exist $i_0,j_0\in\{1,2,\dots,k\}$ $(i_0\neq j_0)$ such that $E=[\mathbf{a}_{i_0},\mathbf{a}_{j_0}]$, but not every segment $[\mathbf{a}_i,\mathbf{a}_j]$ is an edge of $\mathcal{P}$.

Definition 1. Let $\alpha_i,\beta_i\in\mathbb{R}$ be given and $\alpha_i<\beta_i$ $(i=1,2,\dots,n)$. The set

$$Q=\{\mathbf{q}=(q_1,q_2,\dots,q_n)^T : \alpha_i\le q_i\le\beta_i,\ i=1,2,\dots,n\} \tag{1}$$

is called a box (parallelepiped) in $\mathbb{R}^n$.

The box (1) has $n2^{n-1}$ edges. For each edge of the box (1) there exists an index $m\in\{1,2,\dots,n\}$ such that this edge has the form:

$$\{\mathbf{q}=(q_1,q_2,\dots,q_n)^T\in\mathrm{R}^n :\ q_i=\alpha_i \text{ or } q_i=\beta_i \text{ for } i\neq m, \text{ and } \alpha_m\le q_m\le\beta_m\}.$$

The family of edges of Q can be partitioned into n subfamilies $\mathcal{E}_1, \mathcal{E}_2, \dots, \mathcal{E}_n$, where all segments in $\mathcal{E}_i$ $(i = 1, 2, \dots, n)$ are parallel and has the same length. The subfamily $\mathcal{E}_i$ consists of all edges of the box Q of the form

$$\{\mathbf{q} = (q_1, q_2, \dots, q_n)^T : \ q_i \in [\alpha_i, \beta_i], \ q_j = \alpha_j \text{ or } q_j = \beta_j \text{ for } j \neq i\}$$

and all segments in $\mathcal{E}_i$ have the same direction vector $(0, \dots, 0, \beta_i - \alpha_i, 0, \dots, 0)^T$. For simplicity we will consider only the first subfamily $\mathcal{E}_1$. By definition $\mathcal{E}_1$ consists of the segments

$$\{\mathbf{q} = (q_1, q_2, \dots, q_n)^T : \ \alpha_1 \le q_1 \le \beta_1 \text{ and } q_i = \alpha_i \text{ or } q_i = \beta_i \ (i = 2, 3, \dots, n)\}.$$

Assume that $\mathcal{E}_1$ consists of the line segments $l_1, l_2, \dots, l_r$ (where $r = 2^{n-1}$). Since the set $l_1 \cup l_2 \cup \cdots \cup l_r$ contains all extremal points of Q (Fig. 1.) then

$$Q = \text{conv}\{l_1 \cup l_2 \cup \cdots \cup l_r\}.$$

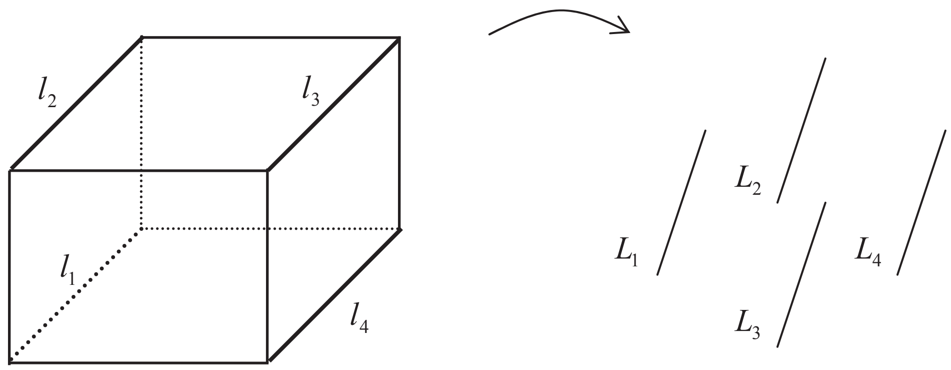

Figure 1. The first subfamily $\mathcal{E}_1 = \{l_1, l_2, l_3, l_4\}$ of Q $(n = 3)$.

Let A be an m by n matrix and $\mathbf{d}$ be an m-vector. Consider the polytope

$$\Pi = AQ + \mathbf{d} = \{A\mathbf{q} + \mathbf{d} : \ \mathbf{q} \in Q\}.$$

Every extreme point in Π is the image of some extreme point of Q and every edge point in Π is the image of some edge point of Q (Ziegler (2006) p. 196).

As pointed out above this manuscript addresses two points:

1) Determination of the edges of the polytope $\mathcal{P} = \text{conv}\{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_k\}$. It is shown that this problem is equivalent to the standard linear programming problem (Theorem 1).

2) Determination of the edges of the polytope $\Pi = AQ + \mathbf{d}$ (where Q is a box, A is a given matrix, $\mathbf{d}$ is a given vector). We show that if $l = [\mathbf{e}_1, \mathbf{e}_2]$ is an edge of Q and the images of $\mathbf{e}_1$ and $\mathbf{e}_2$ are extreme points of Π then the image of $[\mathbf{e}_1, \mathbf{e}_2]$ is an edge of Π (Theorem 4). Here we also give a simple sufficient condition on the matrix A (Theorem 5) which guarantees that all edges of the polytope Π are covered by Theorem 4.

At the end of the sections we give illustrative examples.

2. EDGES OF THE POLYTOPE $\mathcal{P}$

Consider the polytope $\mathcal{P} = \text{conv}\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k\}$. In this section we determine the edges of $\mathcal{P}$.

The problem of efficiently determination of extremal points of a polytope has been studied in many works (see Dulá et al. (1992), Dulá and Helgason (1996), Murty (2009), Rosen et al. (1992), Wallace and Wets (1995), Wets and Witzgall (1967) and references therein). As far as the edges are concerned the following result is obtained in Murty (2009):

Let each points $\mathbf{a}_i$ $(i = 1,2,...,k)$ be an extreme point of $\mathcal{P}$. Then the line segment $[\mathbf{a}_1, \mathbf{a}_2]$ is an edge of $\mathcal{P}$ iff there exists a $\mathbf{c} = (c_1, c_2,..., c_n)^T$ satisfying

$$\mathbf{c}^T(\mathbf{a}_1 - \mathbf{a}_2) = 0$$
$$\mathbf{c}^T(\mathbf{a}_1 - \mathbf{a}_i) > 0 \quad \text{for all} \quad i = 3 \text{ to } k.$$

In this section we show that the problem when $[\mathbf{a}_1, \mathbf{a}_2]$ is an edge of $\mathcal{P}$ is equivalent to a standard linear programming problem of the type $A\mathbf{x} = \mathbf{b}$, $\mathbf{x} \geq 0$, $\mathbf{d}^T\mathbf{c} \to \min$.

On the other hand in Theorem 1 there is no requirement that all $\mathbf{a}_i$ are extreme points.

Let three vectors $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z} \in \mathbb{R}^n$ be given. It is easy to check whether the point $\mathbf{x}$ belongs to the straight line passing from $\mathbf{y}$ and $\mathbf{z}$ or not. Indeed, consider the vectors $\mathbf{y} - \mathbf{x}$ and $\mathbf{z} - \mathbf{x}$. Then $\mathbf{x}$ belongs to the straight line passing from $\mathbf{y}$ and $\mathbf{z}$ if and only if the vectors $\mathbf{y} - \mathbf{x}$ and $\mathbf{z} - \mathbf{x}$ are proportional.

Theorem 1. Let the polytope $\mathcal{P} = \text{conv}\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k\} \subset \mathbb{R}^n$ be given. Consider the line segment $[\mathbf{a}_1, \mathbf{a}_2]$ and its midpoint $\mathbf{b} = \dfrac{\mathbf{a}_1 + \mathbf{a}_2}{2}$. Assume that for every $i \in \{3,4,...,k\}$ the point $\mathbf{a}_i$ does not belong to the straight line passing from the points $\mathbf{a}_1$ and $\mathbf{a}_2$. Consider the following standard linear programming problem (LP) on variables $\lambda_1, \lambda_2, ..., \lambda_k$:

$$(LP): \begin{cases} \lambda_1\mathbf{a}_1 + \lambda_2\mathbf{a}_2 + \cdots + \lambda_k\mathbf{a}_k = \mathbf{b} \\ \lambda_1 + \lambda_2 + \cdots + \lambda_k = 1 \\ \lambda_i \geq 0 \quad (i = 1,2,\ldots,k) \\ \lambda_1 + \lambda_2 \to \min \end{cases} \tag{2}$$

Then the segment $[\mathbf{a}_1, \mathbf{a}_2]$ is an edge of $\mathcal{P}$ if and only if the optimal value of the problem (LP) is equal to 1.

Proof. $\Rightarrow$): Let $[\mathbf{a}_1, \mathbf{a}_2]$ be an edge of the polytope $\mathcal{P}$. Assume that the optimal value of the problem (LP) is strictly less than 1 (by (2) this value is not greater than 1). Then there exist $\lambda_1, \lambda_2, ..., \lambda_k$ such that

$$\begin{aligned} &\lambda_1 + \lambda_2 < 1, \\ &\lambda_1 + \lambda_2 + \cdots + \lambda_k = 1, \\ &\mathbf{b} = \lambda_1 \mathbf{a}_1 + \lambda_2 \mathbf{a}_2 + \cdots + \lambda_k \mathbf{a}_k \\ &\lambda_i \geq 0 \quad (i = 1,2,...,k). \end{aligned} \tag{3}$$

From this it follows that at least one of $\lambda_3, \lambda_4, ..., \lambda_k$ is positive. For simplicity assume that $\lambda_3 > 0$. Obviously $\lambda_3 < 1$ (otherwise $\mathbf{b} = \mathbf{a}_3$, which contradicts the assumption).

Consider the vector $\widetilde{\mathbf{b}}$ on the straight line passing from $\mathbf{a}_3$ and $\mathbf{b}$:

$$\widetilde{\mathbf{b}} = \mathbf{a}_3 + t(\mathbf{b} - \mathbf{a}_3) \tag{4}$$

Where $1 < t < \dfrac{1}{1-\lambda_3}$. Then by (3) we have

$$\begin{aligned} \widetilde{\mathbf{b}} &= \mathbf{a}_3 + t(\lambda_1 \mathbf{a}_1 + \lambda_2 \mathbf{a}_2 + \cdots + \lambda_k \mathbf{a}_k - \mathbf{a}_3) \\ &= t\lambda_1 \mathbf{a}_1 + t\lambda_2 \mathbf{a}_2 + [1 + t(\lambda_3 - 1)]\mathbf{a}_3 + t\lambda_4 \mathbf{a}_4 + \cdots + t\lambda_k \mathbf{a}_k \end{aligned} \tag{5}$$

The coefficients of the right-hand side of (5) sum to 1 and are non-negative. Therefore $\widetilde{\mathbf{b}} \in \mathcal{P}$. By assumption $\mathbf{a}_3 \notin [\mathbf{a}_1, \mathbf{a}_2]$. From assumption also follows that $\widetilde{\mathbf{b}} \notin [\mathbf{a}_1, \mathbf{a}_2]$. Indeed, if $\widetilde{\mathbf{b}} \in [\mathbf{a}_1, \mathbf{a}_2]$ then there exists $\alpha \in [0,1]$ such that $\widetilde{\mathbf{b}} = \alpha\, \mathbf{a}_1 + (1-\alpha)\mathbf{a}_2$. From this and (4) we have

$$\mathbf{a}_3 = \frac{2\alpha - t}{2(1-t)}\mathbf{a}_1 + \frac{2 - 2\alpha - t}{2(1-t)}\mathbf{a}_2$$

which gives that the point $\mathbf{a}_3$ lies on the straight line passing from the points $\mathbf{a}_1$ and $\mathbf{a}_2$. This contradicts the assumption.

Consequently the segment $[\mathbf{a}_3, \widetilde{\mathbf{b}}]$ belongs to $\mathcal{P}$ and has a common point $\mathbf{b}$ with the segment $[\mathbf{a}_1, \mathbf{a}_2]$. This contradicts the assumption that the segment $[\mathbf{a}_1, \mathbf{a}_2]$ is an edge of $\mathcal{P}$.

$\Leftarrow$) : Let the optimal value of the problem (*LP*) be 1. Denote the admissible set of the problem (*LP*), i.e. the set of all $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, ..., \lambda_k)^T \in \mathbb{R}^k$ satisfying (2) by G. By (2) and (3) for all $\boldsymbol{\lambda} \in G$, $\lambda_3 = \lambda_4 = \cdots = \lambda_k = 0$, $\lambda_1 + \lambda_2 = 1$ and $\mathbf{b} = \lambda_1 \mathbf{a}_1 + \lambda_2 \mathbf{a}_2$. By definition of $\mathbf{b}$ we have

$$\lambda_1 \mathbf{a}_1 + \lambda_2 \mathbf{a}_2 = \frac{1}{2}\mathbf{a}_1 + \frac{1}{2}\mathbf{a}_2.$$

This equality together with $\lambda_1 + \lambda_2 = 1$ gives $\lambda_1 = \lambda_2 = \dfrac{1}{2}$. Consequently if the optimal value of (LP) is equal to 1 then the set G contains the unique element $\boldsymbol{\lambda} = (\frac{1}{2}, \frac{1}{2}, 0, ..., 0)^T$ (In other words, the representation

$$\mathbf{b} = \frac{1}{2}\mathbf{a}_1 + \frac{1}{2}\mathbf{a}_2 \tag{6}$$

is the unique representation of $\mathbf{b}$ by $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k$).

By contrary, assume that the segment $[\mathbf{a}_1, \mathbf{a}_2]$ is not an edge of $\mathscr{P}$. Then there exist $\mathbf{p}_1, \mathbf{p}_2 \in \mathscr{P}$, $\mathbf{p}_1 \notin [\mathbf{a}_1, \mathbf{a}_2]$, $\mathbf{p}_2 \notin [\mathbf{a}_1, \mathbf{a}_2]$ and $\mathbf{c} \in [\mathbf{a}_1, \mathbf{a}_2]$ such that $\mathbf{c} \in [\mathbf{p}_1, \mathbf{p}_2]$. Then $\mathbf{b} \in \text{conv}\{\mathbf{a}_1, \mathbf{p}_1, \mathbf{p}_2\}$ or $\mathbf{b} \in \text{conv}\{\mathbf{a}_2, \mathbf{p}_1, \mathbf{p}_2\}$. Without loss of generality assume that $\mathbf{b} \in \text{conv}\{\mathbf{a}_1, \mathbf{p}_1, \mathbf{p}_2\}$. Then there exist α, β and γ such that

$$\begin{gathered} \mathbf{b} = \alpha\, \mathbf{a}_1 + \beta\, \mathbf{p}_1 + \gamma\, \mathbf{p}_2 \\ \alpha + \beta + \gamma = 1, \quad \alpha \geq 0, \ \beta \geq 0, \ \gamma \geq 0. \end{gathered} \tag{7}$$

Since $\mathbf{p}_1, \mathbf{p}_2 \in \mathscr{P}$, $\mathbf{p}_1$ and $\mathbf{p}_2$ can be written as a convex combination of $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k$:

$$\begin{gathered} \mathbf{p}_1 = \sum_{i=1}^{k} \mu_i\, \mathbf{a}_i, \quad \mathbf{p}_2 = \sum_{i=1}^{k} \eta_i\, \mathbf{a}_i \\ \sum_{i=1}^{k} \mu_i = \sum_{i=1}^{k} \eta_i = 1, \quad \mu_i \geq 0, \quad \eta_i \geq 0. \end{gathered} \tag{8}$$

Then by (7)

$$\mathbf{b} = (\alpha + \beta\mu_1 + \gamma\eta_1)\mathbf{a}_1 + (\beta\mu_2 + \gamma\eta_2)\mathbf{a}_2 + \cdots + (\beta\mu_k + \gamma\eta_k)\mathbf{a}_k \tag{9}$$

Comparing (6) and (9) we have

$$\begin{aligned} \alpha + \beta\mu_1 + \gamma\eta_1 &= 1/2 \\ \beta\mu_2 + \gamma\eta_2 &= 1/2 \\ \beta\mu_3 + \gamma\eta_3 &= 0 \\ &\vdots \\ \beta\mu_k + \gamma\eta_k &= 0. \end{aligned} \tag{10}$$

(10) implies

$$\beta\, \mu_3 = \gamma\, \eta_3 = \cdots = \beta\, \mu_k = \gamma\, \eta_k = 0. \tag{11}$$

Second equation of (10) implies $\beta > 0$ or $\gamma > 0$. For simplicity assume that $\beta > 0$. Then by (11)

$$\mu_3 = \cdots = \mu_k = 0. \tag{12}$$

From (8), (12) we have $\mathbf{p}_1 = \mu_1\mathbf{a}_1 + \mu_2\mathbf{a}_2$ ($\mu_1 + \mu_2 = 1, \mu_1 \geq 0, \mu_2 \geq 0$) which contradicts to $\mathbf{p}_1 \notin [\mathbf{a}_1, \mathbf{a}_2]$.

Remark. Theorem 1 establishes the extreme points also, since the set of extreme points is equal to the set of end points of edges.

Example 1. Consider the polytope $\mathscr{P} \subset \mathbb{R}^4$

$$\mathcal{P} = \text{conv}\,\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4, \mathbf{a}_5, \mathbf{a}_6\}$$

Where $\mathbf{a}_1 = (1,2,-2,-5)^T$, $\mathbf{a}_2 = (2,-2,1,7)^T$, $\mathbf{a}_3 = (1,-2,2,7)^T$, $\mathbf{a}_4 = (-2,2,1,-5)^T$, $\mathbf{a}_5 = (2,1,-2,-2)^T$, $\mathbf{a}_6 = (-2,1,2,-2)^T$. The total number of possible segments $[\mathbf{a}_i, \mathbf{a}_j]$ is 15. Application of Theorem 1 and the Simplex Method to each of the segment $[\mathbf{a}_i, \mathbf{a}_j]$ gives that from these segments only 6 segments

$$[\mathbf{a}_1, \mathbf{a}_4],\ [\mathbf{a}_1, \mathbf{a}_5],\ [\mathbf{a}_2, \mathbf{a}_3], [\mathbf{a}_2, \mathbf{a}_5],\ [\mathbf{a}_3, \mathbf{a}_6],\ [\mathbf{a}_4, \mathbf{a}_6]$$

are the edges of the polytope $\mathcal{P}$.

3. EDGES OF A ZONOTOPE

As noted above the edges of a polytope automatically determine all extreme points. In this section for a zonotope (which are important for applications) we show that the extreme points determine the edges. As mentioned above these polytopes have the form

$$\Pi = AQ + \mathbf{d}.$$

Here $Q \subset \mathbb{R}^n$ is the box (1), A is an *m* by *n* matrix, $\mathbf{d}$ is an *m*-vector. If a polytope is shifted by the vector $\mathbf{d}$ then every edge is also shifted by the vector $\mathbf{d}$. Therefore without loss of generality we may assume that $\mathbf{d} = \mathbf{0}$:

$$\Pi = AQ.$$

While zonotopes are well-studied class of polytopes (Fukuda (2004), Grünbaum (2003), Stoer and Witzgall (1970), Ziegler (2006)) the problem of determination of edges has been paid little attention.

It is obvious that $\Pi = \text{conv}\{A\mathbf{a}_1, A\mathbf{a}_2, ..., A\mathbf{a}_k\}$, where $\mathbf{a}_i$ is extreme point of Q (*i*=1, 2, ..., *k*). From now on, we assume that all column vectors of the matrix *A* are nonzero vectors (otherwise we would consider $(n-1)$ dimensional box instead of the $n-$dimensional box (1)).

Let $l_1, l_2, \ldots, l_r$ (where $r = 2^{n-1}$) be edges of the first subfamily $\mathcal{E}_1$ (see Section 1) and let L_s be the image of l_s under transformation *A* ($s = 1, 2, \ldots, r$):

$$L_s = Al_s.$$

The segments L_s are parallel and has the same length (Fig. 1). It can be easily seen that

$$L_s = \{t\mathbf{a} + \mathbf{b}_s :\ \alpha_1 \le t \le \beta_1\} \tag{13}$$

where nonzero $\mathbf{a}$ is the first column vector of the matrix *A*, $\mathbf{b}_s$ is a linear combination of remaining columns (i.e. 2nd, 3th, ..., *n*th columns) of *A* and the coefficients in these linear combinations are α_i or β_i ($i \ne 1$).

Lemma 2. $\Pi = \text{conv}\{L_1 \cup L_2 \cup \cdots \cup \mathrm{L}_r\}$.

Proof. Let $\mathbf{y}$ be any extremal point of Π. Then there exists an extremal point $\mathbf{x}$ of Q such that $\mathbf{x} \in l_s$ and $\mathbf{y} = A\mathbf{x} \in Al_s = L_s$ for some $s \in \{1, 2, \ldots, r\}$. Thus every extremal point of Π is contained in $L_1 \cup L_2 \cup \cdots \cup \mathrm{L}_r$.

Lemma 3. $\Pi = \mathcal{A} + B$ where $\mathcal{A} = \{t\mathbf{a} : \alpha_1 \le t \le \beta_1\}$, $B = \text{conv}\{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_r\}$.
Proof. $L_s = \{t\mathbf{a} + \mathbf{b}_s : \alpha_1 \le t \le \beta_1\} = \mathcal{A} + \mathbf{b}_s \subset \mathcal{A} + B$. Then

$$L_1 \cup L_2 \cup \cdots \cup \mathrm{L}_\mathrm{r} \subset \mathcal{A} + B$$

and

$$\Pi = \text{conv}\{L_1 \cup L_2 \cup \cdots \cup \mathrm{L}_\mathrm{r}\} \subset \mathcal{A} + B .$$

Conversely, if $\mathbf{x} \in \mathcal{A} + B$ then there exist $t_* \in [\alpha_1, \beta_1]$, $\mathbf{b}_s \in B$, $\lambda_s \ge 0$ ($s = 1, 2, \ldots, r$) such that $\lambda_1 + \lambda_2 + \cdots + \lambda_r = 1$ and

$$\begin{aligned} \mathbf{x} &= t_*\mathbf{a} + \lambda_1\mathbf{b}_1 + \cdots + \lambda_r\mathbf{b}_r \\ &= \lambda_1(t_*\mathbf{a} + \mathbf{b}_1) + \cdots + \lambda_r(t_*\mathbf{a} + \mathbf{b}_r). \end{aligned}$$

Since

$$t_*\mathbf{a} + \mathbf{b}_s \in L_s \subset L_1 \cup L_2 \cup \cdots \cup L_r$$

then

$$\mathbf{x} \in \text{conv}\{L_1 \cup L_2 \cup \cdots \cup L_r\} = \Pi.$$

Theorem 4. Let $l = [\mathbf{e}_1, \mathbf{e}_2]$ be an edge of *Q* (i.e. $\mathbf{e}_1$ and $\mathbf{e}_2$ are adjacent extreme points of *Q*). Let $A\mathbf{e}_1$ and $A\mathbf{e}_2$ be extreme points of Π. Then the segment $L = Al = \{A\mathbf{x} : \ \mathbf{x} \in l\}$ is an edge of Π.

Proof. Without loss of generality assume that the segment *l* belongs to the first subfamily $\mathcal{E}_1$ see Section 1). By contrary assume that *L* is not an edge of Π. Then there exist $\mathbf{p}_1, \mathbf{p}_2 \in \Pi$, $\mathbf{p}_1 \notin L$, $\mathbf{p}_2 \notin L$ such that

$$[\mathbf{p}_1, \mathbf{p}_2] \cap L \ne \phi. \tag{14}$$

Since *l* belongs to the first subfamily $\mathcal{E}_1$ the segment *L* is one of L_1, L_2, …, L_r and by (13)

$$L = \{t\mathbf{a} + \mathbf{b} : \quad \alpha_1 \le t \le \beta_1\}$$

where the nonzero vector $\mathbf{a}$ is the first column vector of the matrix *A* and $\mathbf{b} \in \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_r\}$. From (14) it follows that there exist $\tilde{t}_1, \tilde{t}_2, \tilde{t}_3 \in [\alpha_1, \beta_1]$ and $\tilde{\mathbf{b}}_1, \tilde{\mathbf{b}}_2 \in \mathrm{B}$ such that

$$\tilde{t}_1\mathbf{a} + \mathbf{b} \in [\mathbf{p}_1, \mathbf{p}_2] = [\tilde{t}_2\mathbf{a} + \tilde{\mathbf{b}}_1, \tilde{t}_3\mathbf{a} + \tilde{\mathbf{b}}_2]$$

or there exists $\lambda \in [0,1]$ such that

$$\tilde{t}_1\mathbf{a} + \mathbf{b} = \tilde{t}_2\mathbf{a} + \tilde{\mathbf{b}}_1 + \lambda(\tilde{t}_3\mathbf{a} - \tilde{t}_2\mathbf{a} + \tilde{\mathbf{b}}_2 - \tilde{\mathbf{b}}_1). \tag{15}$$

Denote $t_0 = -\tilde{t}_1 + (1-\lambda)\tilde{t}_2 + \lambda\tilde{t}_3$. Then

$$t_0 \in [-(\beta_1 - \alpha_1), (\beta_1 - \alpha_1)].$$

The equality (15) can be written as

$$\mathbf{b} = t_0\mathbf{a} + (1-\lambda)\widetilde{\mathbf{b}}_1 + \lambda\widetilde{\mathbf{b}}_2. \tag{16}$$

Consider three cases:

Case 1. $t_0 > 0$.

Using (16) the following identity can be written:

$$\alpha_1\mathbf{a} + \mathbf{b} = \frac{t_0}{t_0 + (\beta_1 - \alpha_1)}(\beta_1\mathbf{a} + \mathbf{b}) + \frac{\beta_1 - \alpha_1}{t_0 + (\beta_1 - \alpha_1)}[(1-\lambda)(\alpha_1\mathbf{a} + \tilde{\mathbf{b}}_1) + \lambda(\alpha_1\mathbf{a} + \tilde{\mathbf{b}}_2)].$$

By assumption, the point $\alpha_1\mathbf{a} + \widetilde{\mathbf{b}}_2$, which is one of the end points of the segment L, is an extreme point of Π. By Lemma 3, $\alpha_1\mathbf{a} + \widetilde{\mathbf{b}}_1 \in \Pi$ and $\alpha_1\mathbf{a} + \widetilde{\mathbf{b}}_2 \in \Pi$. Therefore

$$(1-\lambda)(\alpha_1\mathbf{a} + \widetilde{\mathbf{b}}_1) + \lambda(\alpha_1\mathbf{a} + \widetilde{\mathbf{b}}_2) \in \Pi.$$

From assumptions it follows that

$$(\beta_1\mathbf{a} + \mathbf{b}) \neq (1-\lambda)(\alpha_1\mathbf{a} + \widetilde{\mathbf{b}}_1) + \lambda(\alpha_1\mathbf{a} + \widetilde{\mathbf{b}}_2),$$

otherwise $[t_0 + (\beta_1 - \alpha_1)]\mathbf{a} = \mathbf{0}$ and since $\mathbf{a} \neq \mathbf{0}$ then $t_0 = \alpha_1 - \beta_1 < 0$ and this contradicts to the assumption $t_0 > 0$. Thus the extreme point $\alpha_1\mathbf{a} + \mathbf{b}$ is expressed as a convex combination of two different points from Π (recall that $\dfrac{t_0}{t_0 + (\beta_1 - \alpha_1)} > 0$). This contradicts to the definition of an extreme point.

Case 2. $t_0 < 0$.

Using the identity

$$\beta_1\mathbf{a} + \mathbf{b} = \frac{-t_0}{(\beta_1 - \alpha_1) - t_0}(\alpha_1\mathbf{a} + \mathbf{b}) + \frac{(\beta_1 - \alpha_1)}{(\beta_1 - \alpha_1) - t_0}[(1-\lambda)(\beta_1\mathbf{a} + \tilde{\mathbf{b}}_1) + \lambda(\beta_1\mathbf{a} + \tilde{\mathbf{b}}_2)]$$

we obtain a contradiction as in the Case 1.

Case 3. $t_0 = 0$.

By (16) $\mathbf{b} = (1-\lambda)\widetilde{\mathbf{b}}_1 + \lambda\widetilde{\mathbf{b}}_2$. Then $\lambda \in (0,1)$. Indeed if $\lambda = 0$ then $\mathbf{b} = \widetilde{\mathbf{b}}_1$ and from (15) and $\mathbf{a} \neq \mathbf{0}$ it follows that $\widetilde{t}_1 = \widetilde{t}_2$. Therefore

$$\mathbf{p}_1 = \widetilde{t}_2\mathbf{a} + \widetilde{\mathbf{b}}_1 = \widetilde{t}_1\mathbf{a} + \mathbf{b} \in L$$

and this is a contradiction. Similarly it can be shown that $\lambda \neq 1$. Therefore $\lambda \in (0,1)$.

Consider the identity

$$\alpha_1\mathbf{a} + \mathbf{b} = (1-\lambda)(\alpha_1\mathbf{a} + \widetilde{\mathbf{b}}_1) + \lambda(\alpha_1\mathbf{a} + \widetilde{\mathbf{b}}_2).$$

From the assumption $\mathbf{p}_1 \notin L$ it follows that $\alpha_1\mathbf{a} + \widetilde{\mathbf{b}}_1 \neq \alpha_1\mathbf{a} + \widetilde{\mathbf{b}}_2$. Thus the extreme point $\alpha_1\mathbf{a} + \mathbf{b}$ of Π is expressed as a convex combination of the two different points from Π and this contradicts to the definition of an extreme point.

The proof of Theorem 4 is complete.

Now we proceed to the question when all edges of the image $\Pi = AQ$ are covered by Theorem 4. It is well-known from convex analysis that every edge point of the image AQ is the image of some edge point of *Q*. On the other hand the end points of edges are extreme points. From these observations we conclude that for the segment $[\mathbf{y}_1, \mathbf{y}_2]$ to be an edge of Π a necessary condition is that $\mathbf{y}_1, \mathbf{y}_2$ are extreme points and there exist edges $E_1, E_2, \ldots, E_{s_0}$ of the box *Q* such that

$$[\mathbf{y}_1, \mathbf{y}_2] = \bigcup_{i=1}^{s_0}(AE_i). \tag{17}$$

If $s_0 = 1$ in (17), then by Theorem 4 the segment $[\mathbf{y}_1, \mathbf{y}_2]$ is an edge of Π. The following theorem shows that under a simple condition on the matrix *A* the equality (17) is possible only when $s_0 = 1$.

Theorem 5. If no two column vectors of the matrix *A* are proportional then the equality (17) is possibly only when $s_0 = 1$.

Proof. Let $\mathbf{a}^i$ be *i*th column vector of the matrix *A*. Then for an edge $l \in \mathcal{E}_i$, where $\mathcal{E}_i$ is *i*th subfamily of edges of *Q* (see Section 1) the segment Al has direction vector $(\beta_i - \alpha_i)\mathbf{a}^i$.

Therefore for $i, j \in \{1, 2, \ldots, n\}$, $i \neq j$ and $l \in \mathcal{E}_i$, $\widetilde{l} \in \mathcal{E}_j$ the segments Al and $A\widetilde{l}$ are not parallel since the vector $\mathbf{a}^i$ is not proportional to the vector $\mathbf{a}^j$.

Consequently the equality (17) is possible only when $s_0 = 1$.

Corollary. If no two column vectors of *A* are proportional then all edges of the image $\Pi = AQ$ are covered by Theorem 4.

Example 2. Consider the unit box $Q \subset \mathbb{R}^5$

$Q = \{q_1, \ldots, q_5\} : \; 0 \le q_i \le 1, \; (i = 1, \ldots, 5)\}$,

and let the matrix *A* be defined as

$$A = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 & 0 \\ 1 & 2 & 3 & 2 & 1 \end{pmatrix}.$$

The box *Q* has $2^5 = 32$ extreme points and only the images of 18 extreme points of *Q* are extreme points of the image $\Pi = AQ$ (In Fig. 2, non-extremal points of AQ each of which is image of some extremal point of *Q* are illustrated by cross-symbols. The number of such points is $32 - 18 = 14.$). These extreme points of *Q* are the following points:

$$(0,0,0,00)^T \rightarrow (0,0,0)^T,\ (0,0,0,0,1)^T \rightarrow (4,0,1)^T, (0,0,0,1,1)^T \rightarrow (7,1,3)^T,$$
$$(0,0,1,0,0)^T \rightarrow (2,2,3)^T, (0,0,1,1,0)^T \rightarrow (5,3,5)^T,\ (0,0,1,1,1)^T \rightarrow (9,3,6)^T,$$
$$(0,1,1,0,0)^T \rightarrow (3,5,5)^T,\ (0,1,1,1,0)^T \rightarrow (6,6,7)^T, (0,1,1,1,1)^T \rightarrow (10,6,8)^T,$$
$$(1,0,0,0,0)^T \rightarrow (0,4,1)^T, (1,0,0,0,1)^T \rightarrow (4,4,2)^T,\ (1,0,0,1,1)^T \rightarrow (7,5,4)^T,$$
$$(1,1,0,0,0)^T \rightarrow (1,7,3)^T,\ (1,1,0,0,1)^T \rightarrow (5,7,4)^T, (1,1,0,1,1)^T \rightarrow (8,8,6)^T,$$
$$(1,1,1,0,0)^T \rightarrow (3,9,6)^T, (1,1,1,1,0)^T \rightarrow (6,10,8)^T,\ (1,1,1,1,1)^T \rightarrow (10,10,9)^T.$$

The matrix A has no proportional columns and consequently by Theorem 4 and Theorem 5 the following segments are the edges of $\Pi = AQ$ and there is no other edge:

$$[(0,0,0)^T,(4,0,1)^T],\quad [(6,6,7)^T,(6,10,8)^T],[(4,0,1)^T,(7,1,3)^T],\quad [(0,4,1)^T,(1,7,3)^T],$$
$$[(7,1,3)^T,(7,5,4)^T],\quad [(7,5,4)^T,(8,8,6)^T],[(5,3,5)^T,(9,3,6)^T],\quad [(5,7,4)^T,(8,8,6)^T],$$
$$[(3,5,5)^T,(6,6,7)^T],\quad [(6,10,8)^T,(10,10,9)^T],[(6,6,7)^T,(10,6,8)^T],\quad [(0,0,0)^T,(0,4,1)^T],$$
$$[(0,4,1)^T,(4,4,2)^T],\quad [(7,1,3)^T,(9,3,6)^T],[(4,4,2)^T,(5,7,4)^T],\quad [(2,2,3)^T,(3,5,5)^T],$$
$$[(1,7,3)^T,(3,9,6)^T],\quad [(9,3,6)^T,(10,6,8)^T],[(3,9,6)^T,(6,10,8)^T],\quad [(0,0,0)^T,(2,2,3)^T],$$
$$[(10,6,8)^T,(10,10,9)^T],\quad [(4,0,1)^T,(4,4,2)^T],[(4,4,2)^T,(7,5,4)^T],\quad [(2,2,3)^T,(5,3,5)^T],$$
$$[(1,7,3)^T,(5,7,4)^T],\quad [(5,3,5)^T,(6,6,7)^T],[(8,8,6)^T,(10,10,9)^T],\quad [(3,3,5)^T,(3,9,6)^T].$$

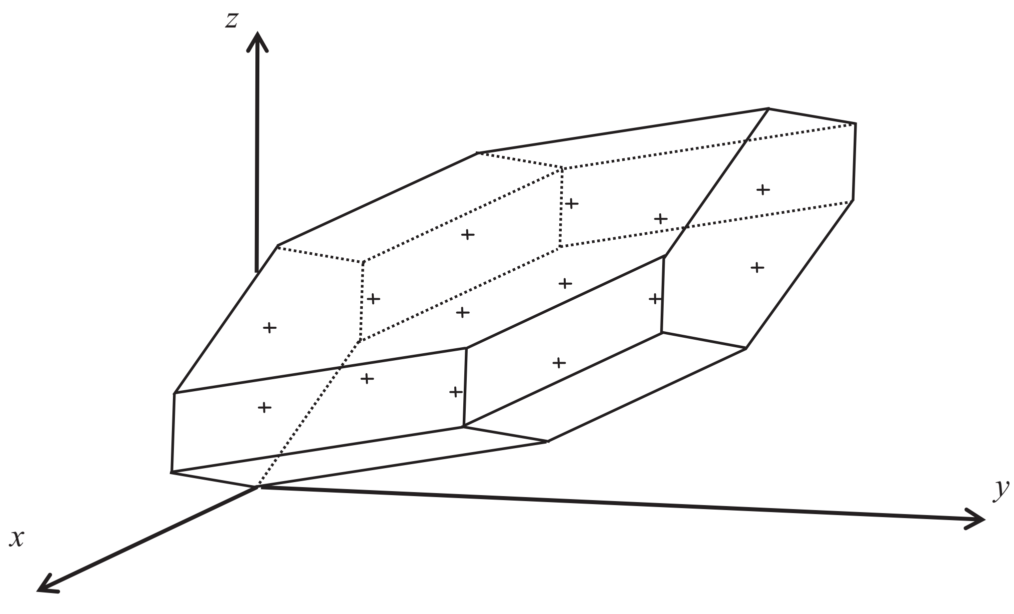

Figure 2. The image of *Q* under the transformation *A*.

The case $s_0 \geq 2$ in (17) is possible only when a), b) and c) below hold simultaneously:

a) At least two column vectors of the matrix *A* are proportional.
b) At least two segments from the different families $A\mathcal{E}_i$ and $A\mathcal{E}_j$ ($i \neq j$) generate a new segment.
c) End points of this new segment are extreme points of $\Pi = AQ$.

This shows that the case $s_0 \geq 2$ in (17) is a very extreme case. Nevertheless if this case occurs then Theorem 1 can be applied.

4. CONCLUSION

In this paper, general polytope which is defined as the convex hull of a finite number of points is considered. It is shown that the line segment connecting two such points is an edge if and only if the optimal value of the corresponding linear programming problem is equal to 1.

For a zonotope, it is shown that the line segment joining the vertex points that are the images of an adjacent vertex of the box Q is an edge of the zonotope.

These results are useful in the application of the Edge Theorem to test stability of a polynomial polytope.

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