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Research Article

SOME NEW INTEGRAL INEQUALITIES FOR n- TIMES DIFFERENTIABLE QUASI-CONVEX FUNCTIONS

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ABSTRACT

In this work, by using an integral identity together with both the Hölder and the Power-Mean integral inequality we establish several new inequalities for *n*-time differentiable quasi-convex functions. Using this inequalities, we obtain some new inequalities connected with means.
Keywords: Convex function, quasi-convex function, Hölder Integral inequality and Power-Mean Integral inequality.
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1. INTRODUCTION

Convexity theory has appeared as a powerful technique to study a wide class of unrelated problems in pure and applied sciences. For some inequalities, generalizations and applications concerning convexity see [7, 11-14]. Recently, in the literature there are so many papers about n-times differentiable functions on several kinds of convexities. In references [4-6, 8, 12, 14, 19-20] readers can find some results about this issue. Many papers have been written by a number of mathematicians concerning inequalities for different classes of quasi -convex functions see for instance the recent papers [1-3, 9, 15-18] and the references within these papers.

Definition 1.1: A function $f: I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex if the inequality

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

is valid for all $x, y \in I$ and $t \in [0,1]$. If this inequality reverses, then f is said to be concave on interval $I \neq \emptyset$. This definition is well known in the literature.

Definition 1.2: A function $f: I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be quasi-convex if the inequality

$$f(tx + (1-t)y) \leq max\{f(x), f(y)\}$$

holds for all $x, y \in I$ and $t \in [0,1]$. Clearly, any convex function is a quasi-convex function. Furthermore, there exist quasi-convex functions which are not convex [10].

Let $0 < a < b$, throughout this pap er we will use

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$$A(a,b)=\frac{a+b}{2},\ L_p(a,b)=\begin{cases}\left(\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right)^{\frac{1}{p}}, p\neq -1,0\\ \frac{b-a}{lnb-lna}\ , \qquad p=-1\\ \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}\ , \qquad p=0.\end{cases}$$

for the arithmetic and generalized logarithmic mean for $a,b>0$ respectively. Furthermore we will use the following notation:

$$M_{n,q}(f)=max\left\{\left|f^{(n)}(a)\right|^q,\left|f^{(n)}(b)\right|^q\right\}.$$

We will use the following Lemma for we obtain the main results [14].

Lemma 1.1: Let $f: I\subseteq\mathbb{R}\to\mathbb{R}$ be n-times differentiable mapping on I° for $n\in\mathbb{N}$ and $f^{(n)}\in L[a,b]$, where $a,b\in I^\circ$ with $a<b$, we have the identity

$$\sum_{k=0}^{n-1}(-1)^k\left(\frac{f^{(k)}(b)b^{k+1}-f^{(k)}(a)a^{k+1}}{(k+1)!}\right)-\int_a^b f(x)dx=\frac{(-1)^{n+1}}{n!}\int_a^b x^n f^{(n)}(x)dx.$$

where an empty sum is understood to be nil.

2. MAIN RESULTS AND THEIR APPLICATIONS

Theorem 2.1. For $\forall n\in\mathbb{N}$; let $f: I\subset(0,\infty)\to\mathbb{R}$ be n-times differentiable function on I° and $a,b\in I^\circ$ with $a<b$. If $\left|f^{(n)}\right|^q$ for $q>1$ is quasi-convex on $[a,b]$, then the following inequality holds:

$$\left|\sum_{k=0}^{n-1}(-1)^k\left(\frac{f^{(k)}(b)b^{k+1}-f^{(k)}(a)a^{k+1}}{(k+1)!}\right)-\int_a^b f(x)dx\right|\leq\frac{1}{n!}(b-a)M_{n,q}^{\frac{1}{q}}(f)L_{np}^n(a,b)$$

Proof. If $\left|f^{(n)}\right|^q$ for $q>1$ is quasi-convex on $[a,b]$, using Lemma1.1, the Hölder integral inequality and

$$\left|f^{(n)}(x)\right|^q=\left|f^{(n)}\left(\frac{x-a}{b-a}b+\frac{b-x}{b-a}a\right)\right|^q\leq M_{n,q}(f),$$

we have

$$\left|\sum_{k=0}^{n-1}(-1)^k\left(\frac{f^{(k)}(b)b^{k+1}-f^{(k)}(a)a^{k+1}}{(k+1)!}\right)-\int_a^b f(x)dx\right|\leq\frac{1}{n!}\left(\int_a^b x^{np}dx\right)^{\frac{1}{p}}\left(\int_a^b\left|f^{(n)}(x)\right|^q dx\right)^{\frac{1}{q}}$$

$$=\frac{1}{n!}\left(\int_a^b x^{np}dx\right)^{\frac{1}{p}}\left(\int_a^b\left|f^{(n)}\left(\frac{x-a}{b-a}b+\frac{b-x}{b-a}a\right)\right|^q dx\right)^{\frac{1}{q}}$$

$$\leq\frac{1}{n!}\left(\int_a^b x^{np}dx\right)^{\frac{1}{p}}\left(\int_a^b M_{n,q}(f)dx\right)^{\frac{1}{q}}$$

$$=\frac{1}{n!}\left(\left.\frac{x^{np+1}}{np+1}\right|_a^b\right)^{\frac{1}{p}}\left(M_{n,q}(f)(b-a)\right)^{\frac{1}{q}}$$

$$=\frac{1}{n!}(b-a)M_{n,q}^{\frac{1}{q}}(f)\left[\frac{b^{np+1}-a^{np+1}}{(np+1)(b-a)}\right]^{\frac{1}{p}}$$

$$= \frac{1}{n!}(b-a)M_{n,q}^{\frac{1}{q}}(f)L_{np}^{n}(a,b)$$

This completes th e proof of theorem.

Corollary 2.1. Under the conditions Theorem 2.1 for $n = 1$ we have the following inequality:

$$\left|\frac{f(b)b - f(a)a}{b-a} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq M_{1,q}^{\frac{1}{q}}(f)L_p(a,b)$$

Proposition 2.1. Let $a, b \in (0,\infty)$ with $a < b$, $p, q > 1, \frac{1}{p} + \frac{1}{q} = 1$ and $m \in \mathbb{Z}\backslash\{-2q, -q\}$, we have the following inequalities:

$$\begin{cases} L_{\frac{m}{q}+1}^{\frac{m}{q}+1}(a,b) \leq b^{\frac{m}{q}}L_p(a,b), \quad for \quad m > 0 \\ L_{\frac{m}{q}+1}^{\frac{m}{q}+1}(a,b) \leq a^{\frac{m}{q}}L_p(a,b), \qquad for \quad m < 0. \end{cases}$$

Proof. Let $f(x) = \frac{q}{m+q}x^{\frac{m}{q}+1}$, $x \in (0,\infty)$. Then $|f'(x)|^q = x^m$ is quasi-convex on $(0,\infty)$ and the result follows directly from Corollary 2.1.

Theorem 2.2. Let $f: (0,\infty) \subset \mathbb{R} \to \mathbb{R}$ be n-times differentiable function and $0 \leq a < b$. If $\left|f^{(n)}\right|^q \in L[a,b]$ and $\left|f^{(n)}\right|^q$ for $q \geq 1$ is quasi-convex on $[a,b]$, then the following inequality holds:

$$\left|\sum_{k=0}^{n-1}(-1)^k\left(\frac{f^{(k)}(b)b^{k+1} - f^{(k)}(a)a^{k+1}}{(k+1)!}\right) - \int_a^b f(x)dx\right| \leq \frac{1}{n!}(b-a)M_{n,q}^{\frac{1}{q}}(f)L_n^n(a,b)$$

where $p = 1 - \frac{1}{q}$ and $p > 1$.

Proof. From Lemma1.1 and Power-mean integral inequality, we obtain

$$\left|\sum_{k=0}^{n-1}(-1)^k\left(\frac{f^{(k)}(b)b^{k+1} - f^{(k)}(a)a^{k+1}}{(k+1)!}\right) - \int_a^b f(x)dx\right|$$

$$\leq \frac{1}{n!}\left(\int_a^b x^n dx\right)^{1-\frac{1}{q}}\left(\int_a^b x^n\left|f^{(n)}(x)\right|^q dx\right)^{\frac{1}{q}}$$

$$= \frac{1}{n!}\left(\int_a^b x^n dx\right)^{1-\frac{1}{q}}\left(\int_a^b x^n\left|f^{(n)}\left(\frac{x-a}{b-a}b + \frac{b-x}{b-a}a\right)\right|^q dx\right)^{\frac{1}{q}}$$

$$\leq \frac{1}{n!}\left(\int_a^b x^n dx\right)^{1-\frac{1}{q}}\left(\int_a^b x^n M_{n,q}(f)dx\right)^{\frac{1}{q}}$$

$$= \frac{1}{n!}(b-a)M_{n,q}^{\frac{1}{q}}(f)L_n^n(a,b).$$

Corollary 2.2. Under the conditions Theorem 2.2 for $n = 1$ we have the following inequality:

$$\left|\frac{f(b)b - f(a)a}{b-a} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq M_{1,q}^{\frac{1}{q}}(f)A(a,b).$$

Proposition 2.2. Let $a, b \in (0,\infty)$ with $a < b$, $q \geq 1$ and $m \in \mathbb{Z}\backslash\{-2q,-q\}$, we have the following inequalities:

$$\begin{cases} L_{\frac{m}{q}+1}^{\frac{m}{q}+1}(a,b) \leq b^{\frac{m}{q}} A(a,b), & for \ m > 0 \\ L_{\frac{m}{q}+1}^{\frac{m}{q}+1}(a,b) \leq a^{\frac{m}{q}} A(a,b), & for \ m < 0 \end{cases}$$

Proof. The result follows directly from Theorem 2.2 for the function $f(x) = \frac{q}{m+q} x^{\frac{m}{q}+1}$, $x \in (0,\infty)$.

Corollary 2.3. Using Proposition 2.2. for $m = 1$, we have following inequalities:

$$L_{1+\frac{1}{q}}^{1+\frac{1}{q}}(a,b) \leq b^{\frac{1}{q}} A(a,b)$$

Corollary 2.4. Using Proposition 2.2. for $q = 1$, we have following inequalities:

$$\begin{cases} L_{m+1}^{m+1}(a,b) \leq b^m A(a,b), & for \ m > 0 \\ L_{m+1}^{m+1}(a,b) \leq a^m A(a,b), & for \ m < 0 \end{cases}$$

Corollary 2.5. Using Corollary 2.4. for $m = 1$, we have following inequality:

$$L_2^2(a,b) \leq bA(a,b).$$

Corollary 2.6. Under the conditions Theorem 2.2 for $q = 1$ we have the following inequality:

$$\left| \sum_{k=0}^{n-1} (-1)^k \left(\frac{f^{(k)}(b) b^{k+1} - f^{(k)}(a) a^{k+1}}{(k+1)!} \right) - \int_a^b f(x)dx \right| \leq \frac{1}{n!}(b-a) M_{n,1}(f) L_n^n(a,b).$$

A generalization of Theorem 2.1. is given as follow:

Theorem 2.3. Let $f: (0,\infty) \subset \mathbb{R} \rightarrow \mathbb{R}$ be n-times differentiable function and $0 \leq a < b$. If $\left|f^{(n)}\right|^q \in L[a,b]$ and $\left|f^{(n)}\right|^q$ for $p, q > 1$, is quasi-convex on $[a,b]$, then the following inequalities holds:

$$\left| \sum_{k=0}^{n-1} (-1)^k \left(\frac{f^{(k)}(b) b^{k+1} - f^{(k)}(a) a^{k+1}}{(k+1)!} \right) - \int_a^b f(x)dx \right| \leq \frac{b-a}{n!} M_{n,q}^{\frac{1}{q}}(f) L_{ip}^i(a,b) L_{(n-i)q}^{n-i}(a,b),$$

where $i = 0,1,2, \dots n$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Proof: If $\left|f^{(n)}\right|^q$ for $q > 1$ is quasi-convex on $[a,b]$, using Lemma1.1 and the Hölder integral inequality, we have the following inequalities respectively:

$$\left| \frac{(-1)^{n+1}}{n!} \int_a^b x^n f^{(n)}(x)dx \right| \leq \frac{1}{n!} \left(\int_a^b 1^p dx \right)^{\frac{1}{p}} \left(\int_a^b x^{nq} \left|f^{(n)}(x)\right|^q dx \right)^{\frac{1}{q}}$$

$$\leq \frac{1}{n!} \left(\int_a^b 1. dx \right)^{\frac{1}{p}} \left(\int_a^b x^{nq} M_{n,q}(f) dx \right)^{\frac{1}{q}}$$

$$= \frac{1}{n!}(b-a) M_{n,q}^{\frac{1}{q}}(f) L_{nq}^n(a,b),$$

$$\left|\frac{(-1)^{n+1}}{n!}\int_a^b x^n f^{(n)}(x)dx\right| \leq \frac{1}{n!}\left(\int_a^b x^p dx\right)^{\frac{1}{p}}\left(\int_a^b x^{(n-1)q}\left|f^{(n)}(x)\right|^q dx\right)^{\frac{1}{q}}$$

$$\leq \frac{1}{n!}\left(\int_a^b x^p dx\right)^{\frac{1}{p}}\left(\int_a^b x^{(n-1)q} M_{n,q}(f)dx\right)^{\frac{1}{q}}$$

$$= \frac{1}{n!}(b-a)M_{n,q}^{\frac{1}{q}}(f)L_p(a,b)L_{(n-1)q}^{n-1}(a,b),$$

$$\vdots$$

$$\left|\frac{(-1)^{n+1}}{n!}\int_a^b x^{n-1}xf^{(n)}(x)dx\right| \leq \frac{1}{n!}\left(\int_a^b x^{(n-1)p}dx\right)^{\frac{1}{p}}\left(\int_a^b x^q\left|f^{(n)}(x)\right|^q dx\right)^{\frac{1}{q}}$$

$$\leq \frac{1}{n!}\left(\int_a^b x^{(n-1)p}dx\right)^{\frac{1}{p}}\left(\int_a^b x^q M_{n,q}(f)dx\right)^{\frac{1}{q}}$$

$$= \frac{1}{n!}(b-a)M_{n,q}^{\frac{1}{q}}(f)L_{(n-1)p}^{n-1}(a,b)L_q(a,b).$$

The proof of case $i = n$ is given in Theorem 2.1.

Corollary 2.7. Under the conditions Theorem 2.3 for $n = 1$ we have the following inequalities respectively:

$$\left|\frac{f(b)b - f(a)a}{b-a} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq M_{1,q}^{\frac{1}{q}}(f)min\{L_q(a,b), L_p(a,b)\},$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proposition 2.3. Let $a, b \in (0,\infty)$ with $a < b$, $q > 1$ and $m \in \mathbb{Z}\backslash\{-2q, -q\}$, we have

$$\begin{cases} L_{\frac{m}{q}+1}^{\frac{m}{q}+1}(a,b) \leq b^{\frac{m}{q}} min\{L_q(a,b), L_p(a,b)\}, & for \quad m > 0 \\ A(a,b) \leq min\{L_q(a,b), L_p(a,b)\}, & for\, m = 0 \\ L_{\frac{m}{q}+1}^{\frac{m}{q}+1}(a,b) \leq a^{\frac{m}{q}} min\{L_q(a,b), L_p(a,b)\}, & for \quad m < 0. \end{cases}$$

Proof. The result follows directly from Corollary 2.7 for the function $f(x) = \frac{q}{m+q}x^{\frac{m}{q}+1}$, $x \in (0,\infty)$.

Corollary 2.8. For $m = 1$ from Proposition 2.3, we obtain the following inequality:

$$L_{\frac{1}{q}+1}^{\frac{1}{q}+1}(a,b) \leq b^{\frac{1}{q}} min\{L_q(a,b), L_p(a,b)\}.$$

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