Sigma J Eng Nat Sci, Vol. 39, No. 2, pp. 203-212, June, 2021

Sigma Journal of Engineering and Natural Sciences –
Sigma Mühendislik ve Fen Bilimleri Dergisi
Web page info: https://sigma.yildiz.edu.tr DOI: 10.14744/sigma.2021.00010

Research Article

Sinc-Galerkin method for solving system of singular perturbed reaction-diffusion problems

Aydin SECER[1,*], Ismail ONDER[1], Muslum OZISIK[1]

[1]*Department of Mathematical Engineering, Yildiz Technical University, Istanbul, Turkey*

ARTICLE INFO

Article history
Received: 04 January 2021
Accepted: 09 April 2021

Key words:
Sinc-Galerkin method, system of singularly perturbed reaction-diffusion problems.

ABSTRACT

In this paper, the system of singularly Perturbed Reaction-Diffusion problems which are commonly used in physics and chemistry branches of science, were investigated. Sinc-Galerkin Method was used to obtaining the solution of problems. Because of there is no article about Sinc-Galerkin Method related to singularly perturbed Reaction-Diffusion problems in literature, the efficiency of the method was shown via this problem. There are important results that occurred after our research and application. Sinc Galerkin Method which was used in this paper as the main solution method gave better results according to parameter robust method and asymptotical initial value method. The figures and the tables show this competence and low errors.

Cite this article as: Secer A, Onder I, Ozisik M. Sinc-Galerkin method for solving system of singular perturbed reaction-diffusion problems. Sigma J Eng Nat Sci 2021;39(2):203–212.

INTRODUCTION

Mathematicians can model many real-life problems mathematically. They generally use differential equations for modeling. Briefly, electrical systems and circuits, mechanics, population growth, the spread of disease are a few of them. On the other hand, applied mathematics interest physical systems that has no explicit solution. If domain of the problem is too large or the problem has a small parameter then its numerical solution gets harder. In these problems, model restricts the domain to small or sets the parameter to zero. As a result, the singular problem arises.

In this paper, we investigate one of those models, singularly perturbation problems (SPP). SPP's are very common problem types in science and engineering. SPP can occur in the various science field as solid mechanics, fluid dynamics, quantum mechanics, optimal control, chemical reactor, reaction-diffusion process [1]. As you can imagine SPP's can have systems like the system of differential

*__Corresponding author.__
*E-mail address: asecer@yildiz.edu.tr

This paper was recommended for publication in revised form by Section Editor Hasan Koten

Published by Yıldız Technical University Press, İstanbul, Turkey

equations. Systems of SPP are used commonly in electrochemical analysis and population dynamic models. Some actual studies about SPP are as follows; theoretical studies on singularly perturbed Kirchhoff problem [2], investigation of asymptotical convergence on singularly perturbed integro-differential equations [3], producing symmetric collocation scheme for linear SPP with two boundary conditions [4] and developing the numerical scheme for singularly perturbed quasilinear equations [5].

In literature, there are various solving methods for SPP. Parameter-uniform numerical method [6], finite element method [7,8], dual finite element method [9], finite difference method [10,11], 6-point interpolatory subdivision method [12], second-order adaptive grid method [13] and also Sinc-Galerkin method [14]. However, the last reference is about singularly perturbation problems not system of SPP.

Sinc function and Sinc-Galerkin method (SGM) is very common in last decades. Frank Stenger was published an article about solving boundary value problems in 1979 [15], after this main article sinc functions became popular for solving boundary value problems (BVP) numerically. SGM is very effective way for solving linear, non-linear, ODE and PDE. Researchers can find articles which uses Sinc-Galerkin method for linear singular BVP [16], nonlinear BVP [17], hyperbolic PDE [18], parabolic PDE [19], fractional [20] equations. Also can be found the solution of Troesch [21], Euler-Bernoulli [22], Bratu [23] and Schrödinger [24] equations as phenomena. Besides the citations given, there are also up-to-date studies on SGM; solution of fourth-order partial integro-differential equation [25], solution of fractional convection-diffusion equation [26], mediated bioelectrocatalysis process [27] and SGM approach for thermal analysis of moving porous fin subject to nanoliquid flow [28]. Considering the citations given, there are quite a lot paper about Sinc-Galerkin Method. Despite all this research, the lack of an article about solution of singularly perturbed reaction-diffusion problems with SGM is the main motivation for this article.

In this paper, solution of the system of singular perturbed reaction-diffusion problems were investigated. Hereby also, the solution of linear and nonlinear system of differential equations with Sinc-Galerkin Method were obtained.

The systems have type which are investigated are

$$\begin{aligned}&P_0(x)u_1(x)+P_1(x)u_1'(x)+P_2(x)u_1''(x)+R_0(x)u_2(x)\\&\quad+R_1(x)u_2'(x)+R_2(x)u_2''(x)=f_1(x)\\[6pt]&Q_0(x)u_1(x)+Q_1(x)u_1'(x)+Q_2(x)u_1''(x)+T_0(x)u_2(x)\\&\quad+T_1(x)u'_2(x)+T_2(x)u_2''(x)=f_2(x)\end{aligned}\tag{1}$$

with boundary conditions as $u_1(a) = u_1(b) = u_2(a) = u_1(b) = 0$ where $a < b < 0$. Thus, problem can be described as System of Singular Perturbed Problems with Dirichlet Type Boundary Conditions.

SINC – BASIS FUNCTION

The function $sinc(z)$ is defined [15] on $\mathbb{C}$ complex plane which $z \in \mathbb{C}$ by

$$sinc(z)=\begin{cases}\dfrac{sin(\pi z)}{\pi z} & z\neq 0\\ 0 & z=0\end{cases}\tag{2}$$

for $h > 0$ and $k = 0, \mp1, \mp2\ldots$ translated sinc functions with evenly spaced nodes are given by [15]

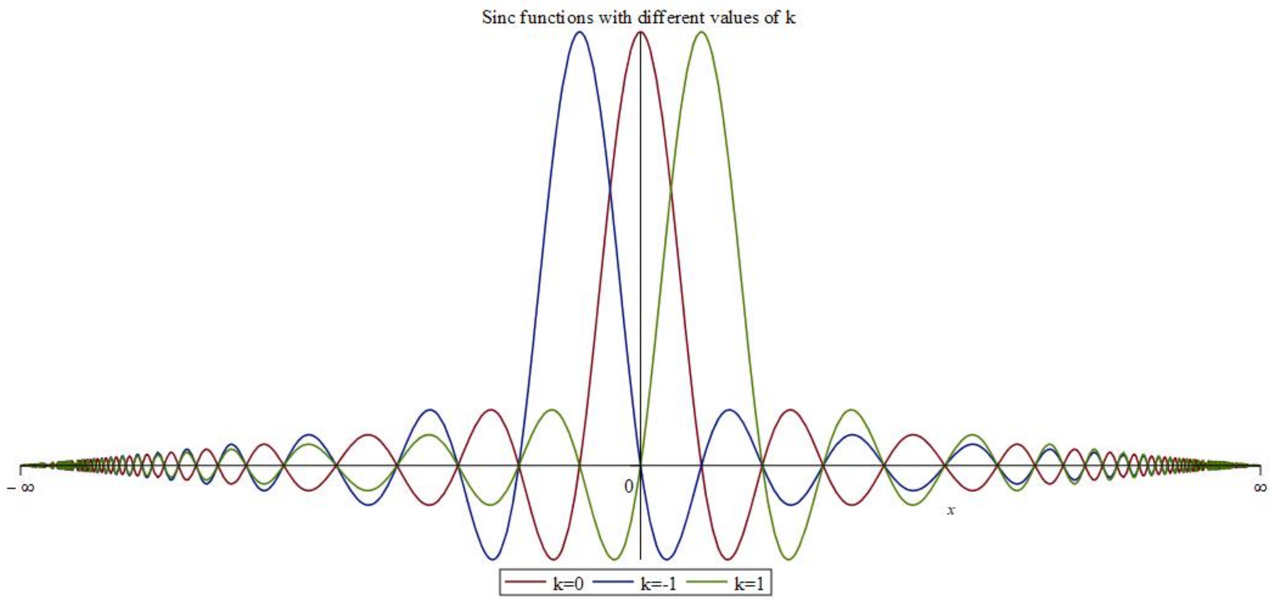

Figure 1. The basis function S(k,h)(x) with h = π/4 [16].

$$S(k,h)(x) = sinc\left(\frac{x-kh}{h}\right)\begin{cases} \dfrac{sin(\pi \frac{x-kh}{h})}{\pi(\frac{x-kh}{h})} & x \neq kh \\ 1 & x = kh \end{cases} \tag{3}$$

The Whittaker Cardinal Function is defined for any $h > 0$ by [15];

$$C(f,h,x) = \sum_{k=-\infty}^{\infty} f(kh) sinc\left(\frac{x-kh}{h}\right), \tag{4}$$

the function converges also. Because of the examination onto the interval (0, 1) in this article, conformal mapping function was selected as

$$\phi = ln\left(\frac{x}{1-x}\right). \tag{5}$$

Interval of problem can be different from (0,1). In this situation, conformal mapping can be obtained as shown in Table 1 [29]. Also evenly, space nodes are denoted by

$$x_k = \phi^{-1}(x) = \frac{e^{kh}}{1+e^{kh}}. \tag{6}$$

This conformal mapping $\phi(x)$ carries domain D_E onto infinite strip D_S. Here D_E and D_S defined as (7) and show in Figure 2.

$$\begin{gathered} D_S \equiv \left\{ w = u + vi,\ |v| < d \leq \frac{\pi}{2} \right\} \\ and \\ D_E \equiv \left\{ x + yi, \left|\arg\left(\frac{z}{1-z}\right)\right| < d \leq \frac{\pi}{2} \right\}. \end{gathered} \tag{7}$$

Table 1. Conformal mapping and nodes for subintervals of $\mathbb{R}$ [16]

Interval		$\phi(x)$	z_k
a	b	$\ln\left(\frac{z-a}{b-z}\right)$	$\frac{a+be^{kh}}{1+e^{kh}}$
0	1	$\ln\left(\frac{z}{1-z}\right)$	$\frac{e^{kh}}{1+e^{kh}}$
0	∞	$\ln(z)$	e^{kh}
0	∞	$\ln(sinh(z))$	$(e^{kh}+\sqrt{e^{2kh}+1})$
$-\infty$	∞	z	kh
$-\infty$	∞	$sinh^{-1}(z)$	kh

Definition 2.1 [14] *Let D_E be a simply connected domain in the complex plane and let ∂D_E be boundaries. Let a,b be these boundary points and $\phi(x)$ be a conformal mapping from D_E onto D_S such that $\phi(a) = -\infty$ and $\phi(b) = \infty$. If the inverse map of $\phi(x)$ is denoted by $\psi(x)$, define*

$$\Gamma \equiv \{\psi(u) \in D_E : -\infty < u < \infty\} \quad z_k = \psi(kh), \\ k = 0, \mp 1, \mp 2 \dots \tag{8}$$

Definition 2.2 [14] *Let $B(D_E)$ be the class of function F that is analytic in D_E and satisfy*

$$\int_{\psi(L+u)} |F(z)|dz \to 0, \ as\ u = \mp\infty, \tag{9}$$

where

$$L \equiv \left\{ iy : |y| < d \leq \frac{\pi}{2} \right\} \text{and satisfy} \\ T(F) = \int_{\partial D_E} |F(z)|\, dz < \infty. \tag{10}$$

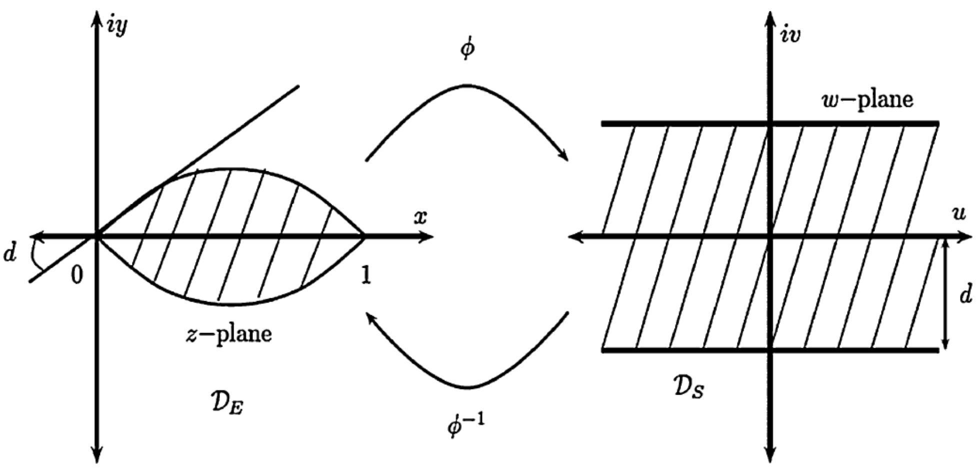

Figure 2. Conformal mapping $\phi(x)$ from eye-shaped domain D_E onto strip domain D_S [16].

Theorem 2.1 [14] *Let Γ be* (0,1), *$F \in B(D_E)$, then for $h > 0$ sufficiently small*

$$\int_\Gamma \left|F(z)\right| dz - h\sum_{j=-\infty}^{\infty} \frac{F(z_j)}{\phi'(z_j)} = \frac{i}{2}\int_{\partial D_E} \frac{F(z)k(\phi,h)(z)}{sin(\pi\phi(z)/h)} dz \equiv I_F, \quad (11)$$

where

$$\left|k(\phi,h)\right|_{z\in\partial D_E} = \left| e^{\left[\frac{i\pi\phi(z)}{h} sgn(Im\phi(z))\right]} \right|_{z\in\partial D_E} = e^{\frac{-\pi d}{h}}. \quad (12)$$

Proof of Theorem 2.1 can be found in [15].

Theorem 2.2 [14] *If there exists positive constants α, β and C such that*

$$\left|\frac{F(x)}{\phi'(x)}\right| \le C \begin{cases} e^{-\alpha|\phi(x)|} & x \in \psi((-\infty,\infty)) \\ e^{-\beta|\phi(x)|} & x \in \psi((0,\infty)) \end{cases} \quad (13)$$

then the error bound for quadrature rule (11) is given by

$$\left|\int_\Gamma \left|F(z)\right| dz - h\sum_{j=-\infty}^{\infty} \frac{F(z_j)}{\phi'(z_j)}\right| \le C\left(\frac{e^{-\alpha Nh}}{\alpha} + \frac{e^{-\beta Nh}}{\beta}\right) + \left|I_F\right|. \quad (14)$$

The infinite sum (11) is truncated with (13) to obtain (14).

Making selections $h = \sqrt{\frac{\pi d}{\alpha N}}$ and $N \equiv [[\frac{\alpha}{\beta}N + 1]]$ where [[.]] is an integer part of thestatement and N is the integer value which specifies the grid size and lastly

$$\int_\Gamma \left|F(x)\right| dx = h\sum_{j=-N}^{N} \frac{F(x_j)}{\phi'(x_j)} + O(e^{-(\pi\alpha dN)^{1/2}}). \quad (15)$$

Theorem 2.1 and 2.2 were used to approximating the integrals that arise in the formulation of discrete systems corresponding to the second order system of singular perturbed boundary value problem.

Theorem 2.3 [14] *Let ϕ be a conformal one-to-one map of simply connected domain D_E onto D_S. Then*

$$\delta_{jk}^0 = \left[S(j,h)\circ\phi(x)\right]\Big|_{x=x_k} = \begin{cases} 1 & k = j \\ 0 & k \ne j \end{cases} \quad (16)$$

$$\delta_{jk}^1 = h\frac{d}{d\phi}\left[S(j,h)\circ\phi(x)\right]\Big|_{x=x_k} = \begin{cases} 1 & k = j \\ \dfrac{(-1)^{k-j}}{k-j} & k \ne j \end{cases} \quad (17)$$

$$\delta_{jk}^2 = h\frac{d^2}{d\phi^2}\left[S(j,h)\circ\phi(x)\right]\Big|_{x=x_k} = \begin{cases} \dfrac{-\pi^2}{3} & k = j \\ \dfrac{-2(-1)^{k-j}}{(k-j)^2} & k \ne j \end{cases} \quad (18)$$

Proof of Theorem (2.2) and Theorem (2.3) can be found in [29].

CONVERGENCE ANALYSIS

Consider the problem in type (1) with Dirichlet Boundary Conditions $u_1(a) = u_1(b) = u_2(a) = u_2(b) = 0$ where $a < x < b$ also where P_i, R_i, Q_i and T_i for $i = 0,1,2$ are analytic on D_E. Sinc approximation was considered by the formula of Whittaker

$$u_1(x) \approx u_{1N}(x) = \sum_{k=-N}^{N} c_k S(k,h)\circ\phi(x)$$

and

$$u_2(x) \approx u_{2N}(x) = \sum_{k=-N}^{N} d_k S(k,h)\circ\phi(x) \quad (19)$$

with

$$S(k,h)(x) = \frac{sin\left(\pi\frac{x-kh}{h}\right)}{\pi\frac{x-kh}{h}}.$$

The unknown coefficients in equation (19) are determined by orthogonalizing the residual with respect to the sinc basis functions. The Galerkin method enables to determine the c_k and d_k coefficients by solving the system of equations

$$\langle Lu_{1_N} - f_1, S(k,h)\circ\phi(x)\rangle = 0$$

and

$$\langle Lu_{2_N} - f_2, S(k,h)\circ\phi(x)\rangle = 0 \;\; (k = -N\ldots0\ldots N). \quad (20)$$

Let $f(x)$ and $g(x)$ be arbitrary functions on D_E. The inner product of these two function is defined as follows:

$$\langle f(x), g(x)\rangle = \int_\Gamma w(x)f(x)g(x)dx, \quad (21)$$

here $w(x)$ is weight function. It's convenient [15] to take $w(x) = \frac{1}{\phi(x)}$.

For equation (1) using the notations (16)–(18) and definition of inner product and using quadrature rule, the followings are obtained

$$\langle P_0(x)u(x), S(k,h)\circ\phi(x)\rangle = \int_\Gamma w(x)P_0(x)u(x)S(k,h)(x)dx \cong h\frac{w(x_k)P_0(x_k)}{\phi'(x_k)}c_k, \quad (22)$$

$$\langle P_1(x)u'(x), S(k,h)\circ\phi(x)\rangle \cong -h\sum_{j=-N}^{N} c_j \times\left[\frac{(w(x_j)P_1(x_j))'}{\phi'(x_j)}\delta_{jk}^0 + \frac{\delta_{jk}^1}{h}\left(w(x_j)P_1(x_j)\right)\right], \quad (23)$$

$$\begin{aligned}&\langle P_2(x)u''(x),S(k,h)\circ\phi(x)\rangle\\&\cong h\sum_{j=-N}^{N}c_j\left[\frac{(w(x_j)P_2(x_j))''}{\phi'(x_j)}\delta_{jk}^0\right.\\&+\frac{\delta_{jk}^1}{h}\left(2(w(x_j)P_2(x_j))'+w(x_j)P_2(x_j)\frac{\phi''(x_j)}{\phi'(x_j)}\right)\\&\left.+w(x_j)P_2(x_j)\phi'(x_j)\frac{\delta_{jk}^2}{h^2}\right].\end{aligned} \tag{24}$$

Here only approximations of P_i was wrote but same formulation can be obtained with changing the function P_i by R_i, Q_i and T_i.

$$\langle f_1(x),S(k,h)\circ\phi(x)\rangle\cong h\frac{w(x_k)f_1(x_k)}{\phi'(x_j)}c_k. \tag{25}$$

Approximation of $f_1(x)$ is equation (25) and same formulation can be obtained for $f_2(x)$. The choices of h and N is same with Theorem (2.2).

These error approximation can be found by using Theorem (2.2) and [29]

- Let v, $w \in B(D)$ for $v=f(x)$ or $p(x)w(x)$ then

$$\begin{aligned}&\left|\int_a^b\left(vw\left[S(j,h)\circ\phi\right]\right)(x)dx-h\left(\frac{vw}{\phi'}\right)(x_j)\right|\\&\le L_0M^{-\frac{1}{2}}e^{(-(\pi\alpha dM)^{1/2})},\end{aligned} \tag{26}$$

where L_0 is positive constant that depends on v,w and d.

- Let $u(p[S(j,h)\circ\phi]w)' \in B(D)$ for

$$\begin{aligned}&\left|\int_a^b\left(pu'[S(j,h)\circ\phi]\right)(x)\,dx+h\sum_{k=-M}^{M}(upw)(x_k)\frac{\delta_{jk}^1}{h}\right.\\&\left.+h\left(\frac{u(pw)'}{\phi'}\right)(x_j)\right|\le L_1M^{\frac{1}{2}}e^{(-(\pi\alpha dM)^{1/2})},\end{aligned} \tag{27}$$

where L_1 is positive constant that depends on u,p,w, ϕ and d.

- Let $u(p[S(j,h)\circ\phi]w)'' \in B(D)$ for

$$\begin{aligned}&\left|\int_a^b\left(pu''[S(j,h)\circ\phi]\right)(x)dx-h\sum_{k=-M}^{M}u(x_k)\right.\\&\left.\times\left[\frac{\delta_j^2k}{h}(pw\phi')(x_k)+\frac{\delta_{jk}^1}{h}\left(\frac{pw\phi''}{\phi'}+2(pw)'\right)(x_j)\right]\right|\\&\le L_2Me^{(-(\pi\alpha dM)^{1/2})},\end{aligned} \tag{28}$$

where L_2 is positive constant that depends on u,p,w, ϕ and d.
Sinc-Galerkin System with (20) becomes

$$\begin{aligned}&0=(Lu_1+Lu_2-f,S(k,h)\circ\phi),\\&0=(P_0u_1,S(j,h)\circ\phi)+(P_1u_1',S(j,h)\circ\phi)\\&\quad-(P_2u_1'',S(j,h)\circ\phi)+(R_0u_2,S(j,h)\circ\phi)\\&\quad+(R_1u_2',S(j,h)\circ\phi)-(R_2u_2'',S(j,h)\circ\phi)\\&\quad-(f_1,S(j,h)\circ\phi).\end{aligned} \tag{29}$$

$$\begin{aligned}&0=(Lu_1+Lu_2-f,S(k,h)\circ\phi),\\&0=(Q_0u_1,S(j,h)\circ\phi)+(Q_1u_1',S(j,h)\circ\phi)\\&\quad-(Q_2u_1'',S(j,h)\circ\phi)\\&\quad+(T_0u_2,S(j,h)\circ\phi)+(T_1u_2',S(j,h)\circ\phi)\\&\quad-(T_2u_2'',S(j,h)\circ\phi)-(f_2,S(j,h)\circ\phi)).\end{aligned} \tag{30}$$

Approximation of Sinc-Galerkin equation is below and bounded by error term is $CMe^{-(\pi\alpha dM)1/2}$;

$$\begin{aligned}&\left|-h\sum_{k=-M}^{M}u_1(x_k)\left[\frac{\delta_{jk}^2}{h}(p_2w\phi')(x_k)+\frac{\delta_{jk}^1}{h}\right.\right.\\&\quad\left.\left.\left(\frac{p_2w\phi''}{\phi'}+2(p_2w)'\right)(x_j)\right]-h\left(\frac{w''u_2p_2}{\phi'}\right)(x_j)\right|\\&\quad+\\&\left|h\sum_{k=-M}^{M}(u_1p_1w)(x_k)\frac{\delta_{jk}^1}{h}+h\left(\frac{u_1(p_1w)'}{\phi'}\right)(x_j)\right.\\&\quad\left.-h\left(\frac{u_1w}{\phi'}\right)(x_j)\right|\\&\quad+\\&\left|-h\sum_{k=-M}^{M}u_2(x_k)\left[\frac{\delta_{jk}^2}{h}(r_2w\phi')(x_k)+\frac{\delta_{jk}^1}{h}\right.\right.\\&\quad\left.\left.\left(\frac{r_2w\phi''}{\phi'}+2(r_2w)'\right)(x_j)\right]-h\left(\frac{w''u_2r_2}{\phi'}\right)(x_j)\right|\\&\quad+\\&\left|h\sum_{k=-M}^{M}(u_2r_1w)(x_k)\frac{\delta_{jk}^1}{h}+h\left(\frac{u_2(r_1w)}{\phi'}\right)(x_j)-h\left(\frac{u_2w}{\phi'}\right)(x_j)\right|\\&+\left|h\left(\frac{f_1}{\phi'}\right)(x_j)\right|+\left|h\left(\frac{f_2}{\phi'}\right)(x_j)\right|\\&\le\left(2L_2+2L_1+4L_0\right)Me^{(-(\pi\alpha dM)^{\frac{1}{2}})}\equiv CMe^{(-(\pi\alpha dM)^{\frac{1}{2}})}.\end{aligned} \tag{31}$$

In literature many of researches uses $Ax = b$ for obtaining the unknown coefficients. Here we suggest that using two times of system with combining unknown coefficients.

$$sys_1=\begin{bmatrix}\langle Lu_{1_{-N}}+Lu_{2_{-N}}-f_1,S(k,h)\circ\phi(x)=0\rangle\\\cdots\\\cdots\\\langle Lu_{1_{+N}}+Lu_{2_{+N}}-f_1,S(k,h)\circ\phi(x)=0\rangle\end{bmatrix}_{mx1}, \tag{32}$$

$$sys_2=\begin{bmatrix}\langle Lu_{1_{-N}}+Lu_{2_{-N}}-f_2,S(k,h)\circ\phi(x)=0\rangle\\\cdots\\\cdots\\\langle Lu_{1_{+N}}+Lu_{2_{+N}}-f_2,S(k,h)\circ\phi(x)=0\rangle\end{bmatrix}_{mx1}. \tag{33}$$

Here sys_1 is obtained from top part of system of differential equation which has functions P_i, R_i and f_1 on the other hand sys_2 is obtained from bottom part of system of differential equation which has functions Q_i, T_i and f_2. Also $m = 2N + 1$. Suggested solution is combining these two systems.

$$sys_3 = \begin{bmatrix} \langle Lu_{1_{-N}} + Lu_{2_{-N}} - f_1, S(k,h) \circ \phi(x) = 0 \rangle \\ \dots \\ \langle Lu_{1_{+N}} + Lu_{2_{+N}} - f_1, S(k,h) \circ \phi(x) = 0 \rangle \\ \langle Lu_{1_{-N}} + Lu_{2_{-N}} - f_2, S(k,h) \circ \phi(x) = 0 \rangle \\ \dots \\ \langle Lu_{1_{+N}} + Lu_{2_{+N}} - f_2, S(k,h) \circ \phi(x) = 0 \rangle \end{bmatrix}_{2mx1} \quad (34)$$

with $vars = [c_{-N} \dots c_0 \dots c_N, d_{-N} \dots d_0 \dots d_N]^T_{1x2m}$.
So if we use $Ax = b$ equation, here formulation turns

$$[sys_3]_{2mx1}.X = [vars]_{2mx1}. \quad (35)$$

We can obtain coefficients c and d. The coefficients will used in Whittaker's Cardinal Function for obtaining approximate solution of $u_1(x)$ and $u_2(x)$ which defined in equation (19).

Table 2. Numerical results of $u_1(x)$ for Example 1 (36) ($N = 32$ and $\varepsilon = 10^{-4}$)

x	Exact Solution	Approximate Solution	Absolute Error
0.1	0.699999399	0.699999975	5.76163E-07
0.11	0.699999854	0.700003458	3.604E-06
0.12	0.699999965	0.700004223	4.25848E-06
0.13	0.699999991	0.700003424	3.43301E-06
0.14	0.699999998	0.700002049	2.05138E-06
0.15	0.7	0.700000736	7.36064E-07
0.16	0.7	0.699999829	1.70418E-07
0.17	0.7	0.699999331	6.6936E-07
0.18	0.7	0.699999224	7.75543E-07
0.19	0.7	0.699999364	6.3551E-07
0.2	0.7	0.699999619	3.81052E-07

NUMERICAL RESULTS

In this chapter of paper, system of SPPs are examined. The equation (35) was solved via Maple.

Example 1

Consider the singularly perturbed system of coupled reaction-diffusion two-point BVPs [30]:

$$\begin{cases} -\varepsilon u_1''(x) + 4u_1(x) - 2u_2(x) = 1 \\ -\varepsilon u_2''(x) - u_1(x) + 3u_2(x) = 2 \\ u_1(0) = u_1(1) = u_2(0) = u_1(1) = 0 \end{cases} . \quad (36)$$

Table 2 shows absolute error for $N = 32$ and $\varepsilon = 10^{-4}$. Also Figure 3 shows exact and approximate graphs of functions with same values. Figure 4 shows error between exact and SGM solution for Example 1 (36). Table 3 shows absolute error of SGM and absolute error of asymptotic initial value method in [30] for Example 1 (36).

Example 2

Consider the constant coefficient problem [11]

$$\begin{cases} -\varepsilon u_1''(x) + 4u_1(x) - 2u_2(x) = 1 \\ -\varepsilon u_2''(x) - u_1(x) + 3u_2(x) = 2 \\ u_1(0) = u_1(1) = u_2(0) = u_1(1) = 0 \end{cases} . \quad (37)$$

In paper [11], there is a table that shows the maximum error of their approximation. They noted the maximum

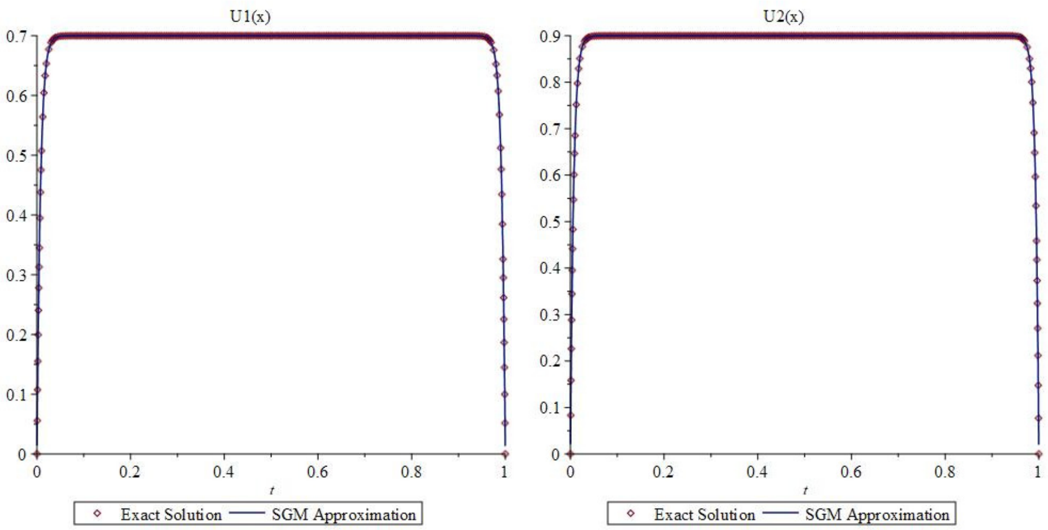

Figure 3. Exact and Approximate solutions of $u_1(x)$ and $u_2(x)$ for Example 1 (36) and $N = 32$ and $\varepsilon = 10^{-4}$.

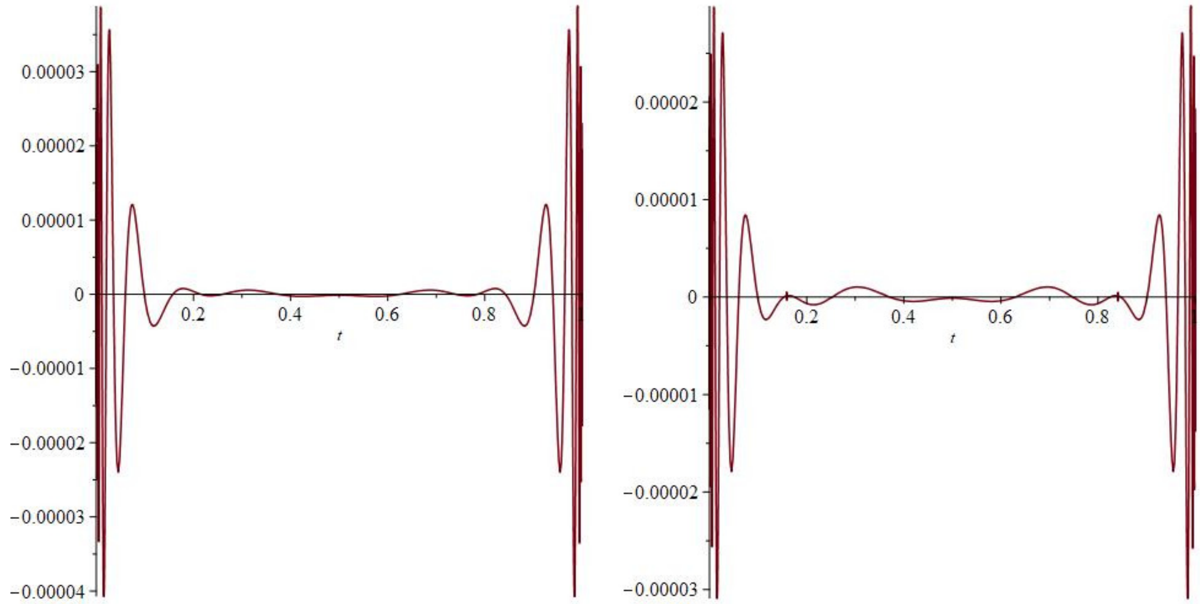

Figure 4. Error between exact solution and approximate solution for Example 1 (36) $N = 32$ and $\varepsilon = 10^{-4}$.

Table 3. Comparison of absolute error of $u_1(x)$ between SGM and [30]

x	Absolute Error of SGM	Absolute Error of [30]
$\sqrt{\varepsilon}$	3,17346E-05	1,82E-02
$2\sqrt{\varepsilon}$	5,8823E-06	6,36E-03
$3\sqrt{\varepsilon}$	2,79308E-05	1,75E-03
$4\sqrt{\varepsilon}$	1,76096E-05	4,48E-04
$5\sqrt{\varepsilon}$	1,93274E-05	1,11E-04
$6\sqrt{\varepsilon}$	1,2414E-07	2,73E-05
$7\sqrt{\varepsilon}$	1,12516E-05	6,68E-06
$8\sqrt{\varepsilon}$	1,06398E-05	1,64E-06
$9\sqrt{\varepsilon}$	4,88453E-06	4,07E-07

error as 0.175908. Figure 5 shows that the graph of exact solution and SGM approximation. Figure 6 shows that the maximum error of the Sinc-Galerkin Method for $u_1(x)$ is around 0.0026 also as seen in Table 4. This means that Sinc-Galerkin Approximation gives a better result for $\varepsilon = 2^{-22}$ according to [11].

Example 3

Consider the constant coefficient problem [30]:

$$\begin{cases} -\varepsilon u_1''(x) + 4u_1(x) - 1u_2(x) = 2 \\ -\varepsilon u_2''(x) - u_1(x) + 3u_2(x) = 3 \\ u_1(0) = u_1(1) = u_2(0) = u_1(1) = 0 \end{cases} . \qquad (38)$$

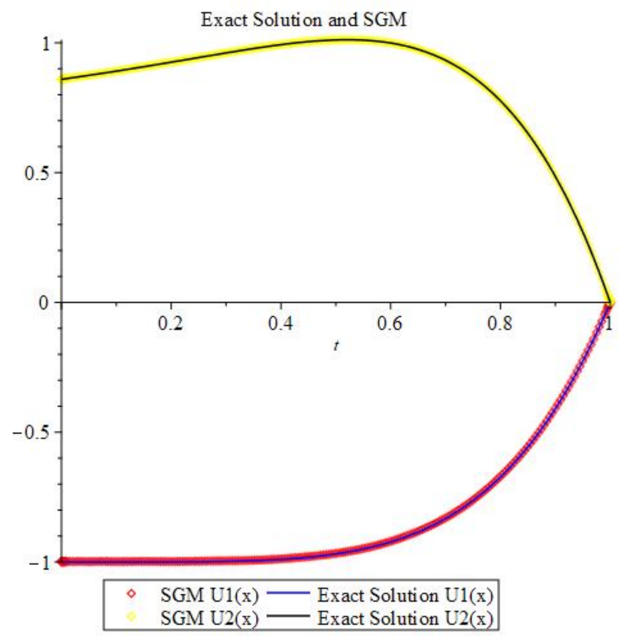

Figure 5. Exact solutions and SGM Approximations for Example 2 (37) $\varepsilon = 2^{-22}$ $N = 64$.

Figure 7 shows that the approximate plots of $u_1(x)$ and $u_2(x)$ for $N = 32$ and $\varepsilon = 10^{-4}$.

Figure 8 shows that maximum error of $u_1(x)$ and $u_2(x)$ is almost 10^{-5}. For getting significant results we will compare with paper [30]. Table 5 shows this comparison.

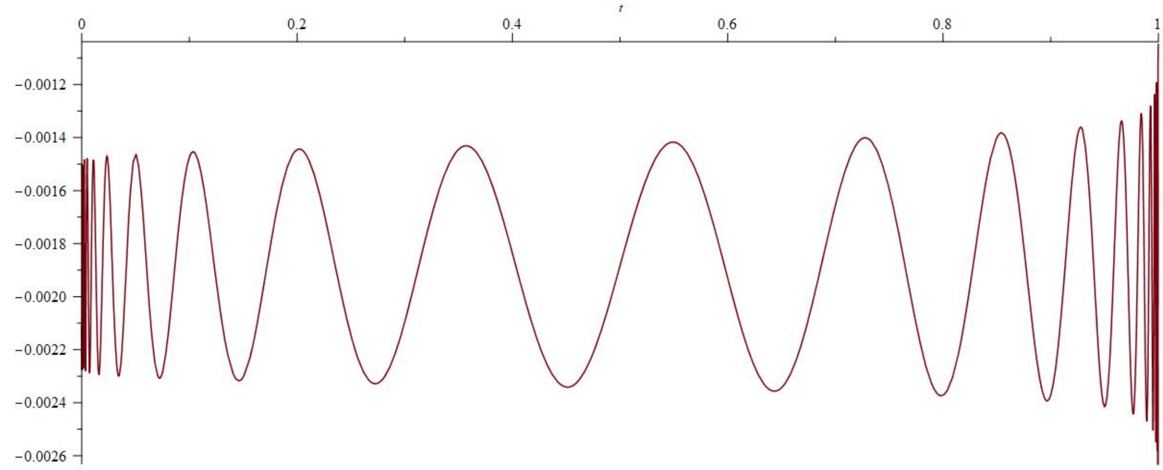

Figure 6. Error between exact solution and SGM Approximation for $u_1(x)$ of Example 2 (37).

Table 4. Numerical Solution of $u_1(x)$ for Example 2 (37) $N = 64$ and $\varepsilon = 2^{-22}$

x	Exact Solution	Approximate Solution	Absolute Error
0,1	−0,99999	−0,998516927	0,001473073
0,11	−0,999983895	−0,998467043	0,001516852
0,12	−0,999975117	−0,998191159	0,001783958
0,13	−0,999962871	−0,997872476	0,002090394
0,14	−0,999946218	−0,997661471	0,002284747
0,15	−0,999924063	−0,997621168	0,002302895
0,16	−0,999895142	−0,997731802	0,002163341
0,17	−0,999858014	−0,997923498	0,001934516
0,18	−0,999811043	−0,99811289	0,001698154
0,19	−0,99975239	−0,998230415	0,001521975
0,2	−0,99968	−0,998234665	0,001445335

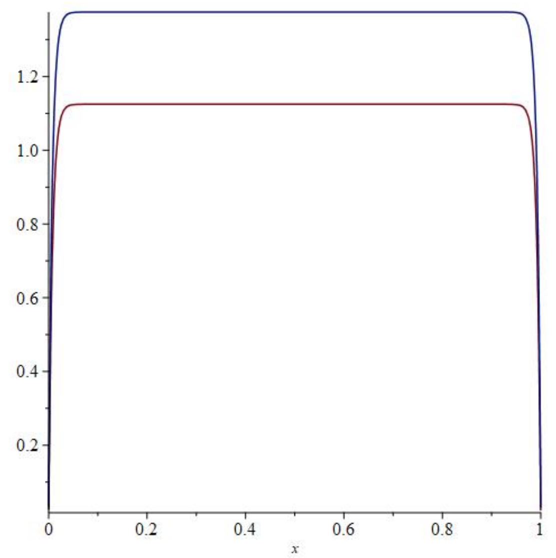

Figure 7. Example 3 (38) $u_1(x)$ and $u_2(x)$ for $N = 32$ and $\varepsilon = 10^{-4}$.

Table 5. Comparison of absolute error of $u_1(x)$ for Example 3 (38) between SGM and [30]

x	Absolute Error of SGM	Absolute Error of [30]
$\sqrt{\varepsilon}$	4.288741E-05	1.347269E-02
$2\sqrt{\varepsilon}$	7.821914E-06	5.098768E-03
$3\sqrt{\varepsilon}$	3.963461E-05	1.486357E-03
$4\sqrt{\varepsilon}$	2.540238E-05	3.947539E-04
$5\sqrt{\varepsilon}$	2.829406E-05	1.004906E-04
$6\sqrt{\varepsilon}$	1.414177E-07	2.504268E-05
$7\sqrt{\varepsilon}$	1.686965E-05	6.171060E-06
$8\sqrt{\varepsilon}$	1.618753E-05	1.511495E-06
$9\sqrt{\varepsilon}$	7.555538E-06	3.689832E-07
$1 - 9\sqrt{\varepsilon}$	4.289042E-05	3.689988E-07
$1 - 8\sqrt{\varepsilon}$	7.823715E-06	1.511527E-06
$1 - 7\sqrt{\varepsilon}$	3.963481E-05	6.171126E-06
$1 - 6\sqrt{\varepsilon}$	2.540091E-05	2.504255E-05
$1 - 5\sqrt{\varepsilon}$	2.829542E-05	1.004889E-04
$1 - 4\sqrt{\varepsilon}$	1.416162E-07	3.947403E-04
$1 - 3\sqrt{\varepsilon}$	1.687000E-05	1.486426E-03
$1 - 2\sqrt{\varepsilon}$	1.618802E-05	5.098954E-03
$1 - \sqrt{\varepsilon}$	7.556071E-06	1.347306E-02

CONCLUSION

In this paper linear systems of singularly perturbed problems are examined. The main purpose is to obtain an approximate solution with Sinc-Galerkin Method. The absolute error of method in Example 1 (36) was calculated approximately as 10^{-7}. In Example 2 (37) was calculated approximately as 10^{-3} and in Example 3 (38) was calculated approximately as 10^{-6}. Considering the low error rate and other studies in the literature, it can be said that the method works properly. Since there is no article about Sinc-Galerkin Method related to SPP's, this paper completes a

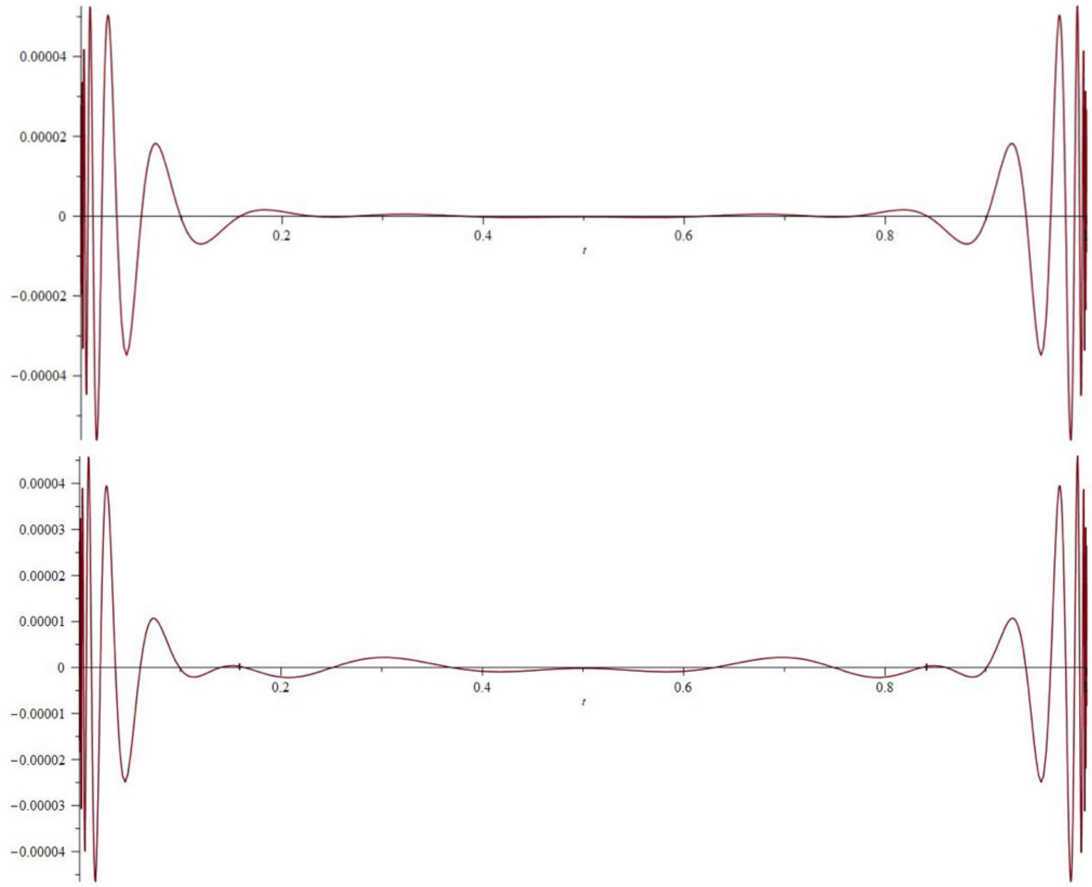

Figure 8. Error between exact solution and SGM Approximation for $u_1(x)$ (above) and $u_2(x)$ (below) of Example 3 (38).

previously untouched area in literature. When compared the effectiveness of Sinc-Galerkin Method with parameter robust method [11] or asymptotical initial value method [30], it's clearly seen that SGM gives better results. In addition to this paper, systems of singularly perturbed nonlinear problems can be investigated as different research.

AUTHORSHIP CONTRIBUTIONS

Concept; A.S.; Design; I.O.; Supervision; A.S.; Materials; A.S.; Data; I.O.; Analysis; I.O.; Literature Search; I.O.; Writing; I.O.; Critical Revision; M.O.

DATA AVAILABILITY STATEMENT

The published publication includes all graphics and data collected or developed during the study.

CONFLICT OF INTEREST

The author declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

ETHICS

There are no ethical issues with the publication of this manuscript.

REFERENCES

[1] Kumar M, Parul. Methods for solving singular perturbation problems arising in science and engineering. Math Comput Model 2011.

[2] Li G, Luo P, Peng S, Wang C, Xiang CL. A singularly perturbed Kirchhoff problem revisited. J Differ Equ 2020; 268: 541–89.

[3] Zhumanazarova A, Cho YI. Asymptotic Convergence of the Solution of a Singularly Perturbed Integro-Differential Boundary Value Problem. Mathematics 2020; 8: 213.

[4] Geng FZ. Piecewise reproducing kernel-based symmetric collocation approach for linear stationary singularly perturbed problems n.d.

[5] Kumar S, Kumar S. High-order convergent methods for singularly perturbed quasilinear problems with integral boundary conditions. Math Methods Appl Sci 2020: mma.6854.

[6] Matthews S, O'Riordan E, Shishkin GI. A numerical method for a system of singularly perturbed reaction-diffusion equations. J Comput Appl Math 2002.
[7] Linß T, Madden N. A finite element analysis of a coupled system of singularly perturbed reaction-diffusion equations. Appl Math Comput 2004.
[8] Lin R, Stynes M. A balanced finite element method for a system of singularly perturbed reaction-diffusion two-point boundary value problems. Numer Algorithms 2015.
[9] Cai Z, Ku J. A dual finite element method for a singularly perturbed reaction-diffusion problem. SIAM J Numer Anal 2020; 58: 1654–73.
[10] Linss T. Analysis of a System of Singularly Perturbed Convection-Diffusion Equations with Strong Coupling. SIAM J Numer Anal 2009.
[11] Bellew S, O'Riordan E. A parameter robust numerical method for a system of two singularly perturbed convection-diffusion equations. Appl Numer Math 2004.
[12] Mustafa G, Ejaz ST, Baleanu D, Ghaffar A, Nisar KS. A subdivision-based approach for singularly perturbed boundary value problem. Adv Differ Equations 2020; 2020: 282.
[13] Cen Z, Liu L Bin, Xu A. A second-order adaptive grid method for a nonlinear singularly perturbed problem with an integral boundary condition. J Comput Appl Math 2021; 385: 113205.
[14] El-Gamel M, Cannon JR. On the solution a of second order singularly-perturbed boundary value problem by the Sinc-Galerkin method. Zeitschrift Fur Angew Math Und Phys 2005.
[15] Stenger F. A "Sinc-Galerkin" Method of Solution of Boundary Value Problems. Math Comput 1979.
[16] Secer A, Kurulay M. The sinc-Galerkin method and its applications on singular Dirichlet-type boundary value problems. Bound Value Probl 2012.
[17] Secer A, Kurulay M, Bayram M, Akinlar MA. An efficient computer application of the sinc-Galerkin approximation for nonlinear boundary value problems. Bound Value Probl 2012.
[18] Secer A. Sinc-Galerkin method for solving hyperbolic partial differential equations. Int J Optim Control Theor Appl 2018.
[19] Secer A. Numerical solution and simulation of second-order parabolic pdes with sinc-galerkin method using maple. Abstr Appl Anal 2013.
[20] Secer A, Alkan S, Akinlar MA, Bayram M. Sinc-Galerkin method for approximate solutions of fractional order boundary value problems. Bound Value Probl 2013.
[21] Zarebnia M, Sajjadian M. The sinc-galerkin method for solving troesch's problem. Math Comput Model 2012.
[22] Smith RC, Bowers KL, Lund J. A fully Sinc-Galerkin method for Euler–Bernoulli beam models. Numer Methods Partial Differ Equ 1992.
[23] Rashidinia J, Maleknejad K, Taheri N. Sinc-Galerkin method for numerical solution of the Bratu's problems. Numer Algorithms 2013.
[24] Dehghan M, Emami-Naeini F. The Sinc-collocation and Sinc-Galerkin methods for solving the two-dimensional Schrödinger equation with nonhomogeneous boundary conditions. Appl Math Model 2013.
[25] Qiu W, Xu D, Guo J. The Crank-Nicolson-type Sinc-Galerkin method for the fourth-order partial integro-differential equation with a weakly singular kernel. Appl Numer Math 2021; 159: 239–58.
[26] Chen LJ, Li MZ, Xu Q. Sinc-Galerkin method for solving the time fractional convection–diffusion equation with variable coefficients. Adv Differ Equations 2020; 2020: 504.
[27] Nabati M, Jalalvand M, Daneh-Dezfuli AR. Solution of mediated bioelectrocatalysis process related to the Michaelis-Menten equation by sinc method. Int J Numer Model Electron Networks, Devices Fields 2020; 33: e2716.
[28] Nabati M, Taherifar S, Jalalvand M. Sinc–Galerkin approach for thermal analysis of moving porous fin subject to nanoliquid flow with different shaped nanoparticles. Math Sci 2021; 1: 3.
[29] Lund J, Bowers KL. Sinc Methods for Quadrature and Differential Equations. 1992.
[30] Valanarasu T, Ramanujam N. An asymptotic initial value method for boundary value problems for a system of singularly perturbed second order ordinary differential equations. Appl Math Comput 2004.