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Quaternionik Lorentz Eğriler Üzerine Bir Çalışma

Araştırma Makalesi / Research Article

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ÖZ

Bu çalışmada öncelikle üç boyutlu Lorentz Uzayda L_Q^3 kuaterniyon ve pseudo-kuaterniyonlar gözönüne alınarak bir uzay-kuaterniyonik eğri için Serret-Frenet Formülleri elde edildi. Daha sonra bunlar kullanılarak bir Kuaterniyonik Lorentz Eğrisi L_Q^4 için Serret-Frenet Formülleri yeniden türetilmiştir.

Anahtar Kelimeler: Regle yüzeyler, Lorentz uzayı, Minkowski uzayı, distribusyon parametresi, Laplacian ve D'Alembert operatörü.

On a Study of the Quaternionic Lorentzian Curve

ABSTRACT

In this study, Serret-Frenet Formulas for a space-quaternionic curve were obtained by considering quaternions and pseudo-quaternions in three-dimensional Lorentz Space L_Q^3. The Serret-Frenet Formulas for a Quaternionic Lorentz Curve L_Q^4 were then re-derived using them.

Keywords: Ruled surfaces, lightlike curves, lightlike surfaces, Minkowski space, distribution parameter, Laplacian and D'Alembertian operator.

1. INTRODUCTION

First of all, some basic definitions and concepts related to the algebra of Lorentzian curve are given. Hence, using this relations, the Serret-Frenet vectors of a Lorentzian curve is rederived. Moreover, some relationships between the Euclidean curve and the Lorentzian curve are obtained.

Definition 1.1 *Let* $\forall x = \sum_{A=1}^{4} x_A \vec{e}_A$, $y = \sum_{A=1}^{4} y_A \ \vec{e}_A \in V$, *be any two element of* V *then, Lie Operation is defined as follows*

$$[x, y] = \sum \left(x_i y_j - x_j y_i\right) \vec{e}_k \quad (1)$$

where, (i, j, k) is the circular permutation of $(1,2,3)$ [1].

Definition 1.2 *Let* S *and* T *be defined as;*

$$Sx = \sum_{i=1}^{3} x_i \vec{e}_i , \quad Tx = x_4 \vec{e}_4$$

for $\quad \forall x = \sum_{A=1}^{4} x_A \vec{e}_A \in V \quad (2)$

It is clear that $ST = TS = 0$ and $S + T = I$ [1].

Definition 1.3 *Let* $\alpha: V \to V$ *be defined as*

$\alpha = -S + T$

a linear transformation, for $\forall x \in V$,

$$\alpha x = -\sum_{i=1}^{3} x_i \vec{e}_i + x_4 \vec{e}_4 = -Sx + Tx. \quad (3)$$

Here if $\alpha^2 = I$ then α is called involutory linear isomorphism [1]. S and T, defined as in (2) are called spatial and temporal projections on V, respectively. α involutory isomorphism which given by (3) is called Hamilton Conjugation on V [1].

Definition 1.4 *The two bilinear forms on* V, *are defined as; for* $\forall x, y \in V$ *then*

$$g(x, y) = \sum_{i=1}^{3} x_i y_i - x_4 y_4, \ h(x, y) = \sum_{A=1}^{4} x_A y_A. \quad (4)$$

In this definition; g and h are non-degenere, real, and symmetric bilinear forms. Furthermore; S and T are defined as in (2) with (3) and they are self-adjoint with respect to bilinear forms defined as in (4).

That is, for each $x, y \in V$,

$$g(Sx, y) = g(x, Sy) = h(x, Sy) = h(Sx, y) \quad (5)$$

$$g(Tx, y) = g(x, Ty) = -h(x, Ty) = -h(Tx, y) \quad (6)$$

$$g(\alpha x, y) = g(x, \alpha y) = -h(x, y) \quad (7)$$

[3].

Definition 1.5 *Let* b *be any symmetric bilinear form on* V. *If, ` $\circ$ ' is a binary operation than define* $\forall x, y \in V$,

$$x \circ y = [x, y] + x_4 Sy + y_4 Sx - b(x, y)\vec{e}_4. \quad (8)$$

In this condition, $(V, \circ)$ is a real algebra (In generally, it is non-associative and non-commutative). For this real algebra, if $b = h$ then the quaternion is called pseudo-

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quaternionic algebra, if $b = g$ then the quaternion is called real-quaternionic algebra on V [1].
In the quaternionic condition; binary operation is denote `$\cdot$ ′ and the pseudo-quaternionic condition is stand for `$*$ ′. Let us consider any $x \in V$, thus $N(x)$ is defined as $x \circ \alpha x \in V$. It is clear that for each $x, y \in V$,

$$N(x) = -b(x, \alpha x)\vec{e}_4. \tag{9}$$

Again, it is clearly seen that,

$$N(x+y) - N(x) - N(y) = x \circ \alpha y + y \circ \alpha x = -\{b(x, \alpha y) + b(y, \alpha x)\}\vec{e}_4. \tag{10}$$

Since α is respect to both g and h self-adjoint then; we have there exist two following equations

$$(x \cdot \alpha y + y \cdot \alpha x) = -2g(x, \alpha y)\vec{e}_4 = 2h(x, y)\vec{e}_4$$

(quaternionic condition)

$$(x * \alpha y + y * \alpha x) = -2h(x, \alpha y)\ e\vec{}\_4 = 2g(x, y)\ e\vec{}\_4$$

(pseudo − qaternionic condition) [2].

Definition 1.6 *Let $x \in V$. The norm of x, $N(x)$ is defined as (9). In the quaternionic condition, $N(x)$ is defined as $h(x,x)\vec{e}_4$. If the another pseudo-quaternionic condition then $N(x)$ is defined as $g(x,x)\vec{e}_4$. Hence; $N(x)$, are used to be, is equal to Euclidean norm $N(x) = \|x\|^2\vec{e}_4$. In the pseudo-quaternionic condition for each $x \in V$, then x is called*

$$g(x,x)\ \rangle\ 0 \quad \text{then space} - \text{like}$$

$$g(x,x) = 0 \quad \text{then null}$$

$$g(x,x)\ \langle\ 0 \quad \text{then time} - \text{like} \tag{11}$$

[2].

Definition 1.7 *A quaternion x is called a unit quaternion if $\|x\|_L = 1$ then, in addition a pseudo quaternion x is unitory whenever $N(x)$ is either $+\vec{e}_4$ or $-\vec{e}_4$ [4].*

Definition 1.8 *If two quaternions x and y (or pseudo-quats) are satisfied $x \cdot \alpha y + y \cdot \alpha x = 0$, $(x * \alpha y + y * \alpha x = 0)$ then they are called ortogonality. Another equvalent condition for ortogonally is*

$$h(x,y) = 0 \ \text{or} \ g(x,y) = 0. \tag{12}$$

Now we define the relationship between quaternionic and pseudo-quaternionic multiplications: Let x and y be two elements of V. By making use of findings (5) and (8) for $x * y - x \cdot y$, we obtain

$$x * y - x \cdot y \equiv \{g(x,y) - h(x,y)\}\vec{e}_4. \tag{13}$$

As a result of equation (13), we have;

$$x * y \equiv x \cdot y - 2x_4y_4\vec{e}_4$$

[1].
In this study L_Q^4 denotes the 4-dimensional pseudo-quaternionic Lorentzian space.

Definition 1.9 *Let $M \subset L_Q^4$ be a curve which has $s-arc$ parameter at Lorentzian space. If the velocity vector of M is $\dot{x}$ then*

$g(\dot{x},\dot{x}) \prec 0$ then $x(s)$ is called time-like curve

$g(\dot{x},\dot{x}) \succ 0$ then $x(s)$ is called space-like curve

$g(\dot{x},\dot{x}) = 0$ then $x(s)$ is called null curve (14)

[4].

Definition 1.10 *Let $M \subset L_Q^4$ be a curve. If the Serret-Frenet frame field is $\{V_1(s), V_2(s), V_3(s), V_4(s)\}$ then the k_i function which is defined as*

$$k_i(s): I \to IR \quad , s \to k_i(s) = \quad g(\ (V_i)'\ (s)\ , V_{i+1}\ (s)\ ) \tag{15}$$

is called, $i-th$ curvature function of M curve and, $k_i(s)$, $1 \le i \le 3$ real number is called $i-th$ curvature of this curve at the point $M(s)$ [4].
Now we study on pseudo-quaternionic Lorentzian Space with the use of these relations.

2. MATERIAL and METHOD

2. 1. The Serret- Frenet formulae of the Pseudo space Quaternionic Lorentzian curve on L_Q^3

Let us show 3-dimensional pseudo-quaternionic Lorentzian space with L_Q^3. Let M be a pseudo-space quaternionic time-like curve.
Let $\tilde{g}$ and L_Q^3 be shown Lorentzian binary operation as, for $\forall x, y \in L_Q^3$

$$\tilde{g}(x,y) = \sum_{i=1}^{2} x_i\ y_i - x_3y_3$$

then, `$\circ$′ binary operation in L_Q^4 is defined as

$$x \circ y = [x,y] + x_4Sy + y_4Sx - b(x,y)\vec{e}_4.$$

If we review the last equation for L_Q^3; $b = \tilde{g}$ and if we consider `$*$′ instead of `$\circ$′ binary operation, we obtain

$$x * y = [x,y] - \tilde{g}(x,y)\vec{e}_4. \tag{16}$$

Let $X: I \to L_Q^3$ be a time like space-quaternionic curve. Hence; $\dot{X} = t$ then, $N(t) = -1$. So, $N(t)$ be defined as

$$N(t) = \tilde{g}(t,t) = t * \alpha t.$$

If we derive this equation with respect to s; we obtain

$$\dot{t} * \alpha\ t + t * \alpha\ \dot{t} = 0. \tag{17}$$

As a result of this;
$\dot{t}$ is $\tilde{g}$-orthogonal to t. That is, $\tilde{g}(\dot{t}, t) = 0$.

$\dot{t} * \alpha$

t is a time-like quaternion.
Hence, we define n_1 space-quaternion and k scalar function as they satisfy the following conditions when $\dot{t}$ is a pseudo-quaternion:

$\dot{\mathrm{t}} = \mathrm{k}\ \mathrm{n}_1\ ,\ \mathrm{k} = \mathrm{N}(\dot{\mathrm{t}}).$ (18)

n_1 is $\tilde{g}$-*orthogonal* to t from (i) there is a n_2 space-quaternion which is satisfy

$\mathrm{t} * \mathrm{n}_1 = \mathrm{n}_2 = -\mathrm{n}_1 * \mathrm{t}.$ (19)

Here, we can write $t * n_2 = -n_1 = -n_2 * t$ and $n_2 * n_1 = -t = -n_1 * n_2$. Hence, t, n_1, n_2 are mutually $\tilde{g}$-*orthogonal* unit pseudo space-quaternion in L_Q^3.

We have derived from (19) and obtained;

$\dot{\mathrm{n}}_2 = (\mathrm{t} * \mathrm{n}_1)',\ \ \dot{\mathrm{n}}_2 = \mathrm{t} * (-\mathrm{kt} + \dot{\mathrm{n}}_1)$ (20)

Thus; $\dot{n}_1 - kt$ is $\tilde{g}$-*orthogonal* to $\dot{t}$ and n_2.

$n_1 = \frac{\ddot{X}}{N(\ddot{X})}$ is unit space-quaternion and

$\dot{n}_1 \in Sp\{t, n_1, n_2\}$.

As a result of this, we can write

$\dot{\mathrm{n}}_1 = \lambda_1 \mathrm{t} + \lambda_2 \mathrm{n}_1 + \lambda_3 \mathrm{n}_2$

Hence; $\|\dot{n}_1\|$ is

$$\tilde{\mathrm{g}}(\dot{\mathrm{n}}_{1,}\dot{\mathrm{n}}_1) = \lambda_1 \tilde{\mathrm{g}}(\mathrm{t}, \dot{\mathrm{n}}_1) + \lambda_2 \tilde{\mathrm{g}}(\mathrm{n}_1, \dot{\mathrm{n}}_1) + \lambda_3 \tilde{\mathrm{g}}(\mathrm{n}_2, \dot{\mathrm{n}}_1),$$
$$\tilde{\mathrm{g}}(\mathrm{t}, \mathrm{t}) = -1.$$

Namely, X is a time-like pseudo-space quaternionic curve. In addition, in a similar way

$$\varepsilon_0 \lambda_1 = \tilde{\mathrm{g}}(\dot{\mathrm{n}}_{1,}\mathrm{t}) = -\tilde{\mathrm{g}}(\dot{\mathrm{t}}, \mathrm{n}_1) = -\mathrm{k}\varepsilon_0\ \ \lambda_1 = -\mathrm{k},$$
$$\frac{1}{\varepsilon_0} = \varepsilon_0$$

$\lambda_1 = -\varepsilon_0 k$ and t is a time like then

$\varepsilon_0 = -1$. Because of; it obtained $\lambda_1 = k$.

In addition, $\lambda_2 = \tilde{g}(\dot{n}_1, n_1) = 0$ and $\lambda_3 = \tilde{\mathrm{g}}(\dot{\mathrm{n}}_1, \mathrm{n}_2) = \varepsilon_0 \mathrm{r}$, hence $\tilde{\mathrm{g}}$ is $\tilde{\mathrm{g}}(\mathrm{n}_1, \mathrm{n}_1) = 1$. Namely, if n_1 is space-like then $\varepsilon_0 = 1$ and $\lambda_3 = \mathrm{r}$. So we obtain

$\dot{\mathrm{n}}_1 = \mathrm{kt} + \mathrm{rn}_2.$ (21)

By substituting (5) in (6), we find

$\dot{\mathrm{n}}_2 = \mathrm{t} * (-\mathrm{kt} + \dot{\mathrm{n}}_1)$

$\dot{\mathrm{n}}_2 = -\mathrm{rn}_1$ (22)

(19), (21) and (22) equations are called Serret-Frenet Formulae of a curve which is time-like pseudo space-quaternionic curve in L_Q^3. $(t, n_1, n_{2,} k, r)$ is called Frenet-Apparatus of this curve.

The matrix form for this Serret-Frenet Apparatus of this curve is given by

$$\begin{bmatrix} \dot{t} \\ \dot{n}_1 \\ \dot{n}_2 \end{bmatrix} = \begin{bmatrix} 0 & k & 0 \\ k & 0 & r \\ 0 & -r & 0 \end{bmatrix} \begin{bmatrix} t \\ n_1 \\ n_2 \end{bmatrix} \quad (23)$$

[4].

2.2. The Serret- Frenet formulae of Pseudo-Quaternionic Lorentzian curve on $\mathbf{L_Q^4}$

Now, by making use of the Serret-Frenet formulae of a pseudo space-quaternionic Lorentzian curve at L_Q^3. We have rederived this formulae for one pseudo-quaternionic curve on L_Q^4: let $\tilde{X} = \sum_{A=1}^{4} q_A(s)\vec{e}_A$ be a time-like curve. The pseudo-quaternionic Lorentzian multiplication is shown by g. We have

$\dot{T} = K\, N_1,\ \ K = N\ (\dot{T}),\ \ \ \ N\ (T) = -1,$

$N\ (N_1) = -1$ (24)

If we derive $N(T) = -1$, then we obtain $g(\dot{T}, T) = 0$. Here by making use of (24) we have

$N_1 * \alpha T + T * \alpha N_1 = 0.$

For a result of them,
N_1 is g-orthogonal to T.
$\mathrm{t} = \mathrm{N}_1 * \alpha \mathrm{T}$ is a space-quaternion.
Here, T and N_1 have unit then t has unit length.
From $t = N_1 * \alpha T$ then, the vector N_1 can choosen as equal to $t * T$ along the curve. Namely this can be written as

$N_1 = t * T.$ (25)

If we derive equation (25) and and use Eqs. (19), (24) and (25), we obtain,

$\dot{N}_1 = \dot{t} * T + t * \dot{T}$

$\dot{N}_1 = K\ T + k\ N_2$ (26)

Here N_2 is

$N_2 = n_1 * T$ (27)

The characterization of N_2 is given as follows:
N_2 is unit.
T, N_1 and N_2 are mutually g-*orthogonal*.
Now, in the derivation of $N_2 = n_1 * T$ by making use of (21), (23) and (24) we have the following result;

$\dot{N}_2 = \dot{n}_1 * T + n_1 * \dot{T}$

$\dot{N}_2 = k\, N_1 + (r - K) N_3$ (28)

Here, N_3 is taken $N_3 = n_2 * T$. According to this condition; the characterization of N_3 is given as follows:
The norm of N_3 is $N(N_3) = 1$.
T,N_1,N_2 and N_3 are mutually g-*orthogonal*.
As a result of these, the derivation of N_3, by making use of (2.0), (24) and (25) we obtain,

$\dot{N}_3 = \dot{n}_2 * T + n_2 * \dot{T}\dot{N}_3 = -(r - K) N_2$ (29)

The equations of (24), (26), (28) and (29) are called The Serret-Frenet Formulae of $\tilde{X}$ time-like pseudo-quaternionic Lorentzian curve at L_Q^3. Thus, $(T, N_1, N_2, N_3, K, k, r - K)$ is called Serret-Frenet Apparatus of this Lorentzian curve.

The matrix form of this Serret-Frenet Apparatus for this Lorentzian curve is given by

$$\begin{bmatrix} \dot{T} \\ \dot{N}_1 \\ \dot{N}_2 \\ \dot{N}_3 \end{bmatrix} = \begin{bmatrix} 0 & K & 0 & 0 \\ K & 0 & k & 0 \\ 0 & k & 0 & r-K \\ 0 & 0 & -(r-K) & 0 \end{bmatrix} \begin{bmatrix} T \\ N_1 \\ N_2 \\ N_3 \end{bmatrix} \qquad (30)$$

[4].

3. CONCLUSIONS

Lorentzian curves have been studied by many mathematicians, but a different study has been done with the terminology of quaternions for a quaternionic Lorentzian curve. Thus the Serret-Frenet formulas of space quaternionic curves are re-derived.

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