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ELASTIC-PLASTIC AND RESIDUAL STRESS ANALYSIS OF AN ALUMINUM DISC UNDER INTERNAL PRESSURES

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ABSTRACT

This paper deals with elastic-plastic stress analysis of a thin aluminum disc under internal pressures. An analytical solution is performed for satisfying elastic-plastic stress-strain relations and boundary conditions for small plastic deformations. The Von-Mises Criterion is used as a yield criterion, and elastic perfectly plastic material is assumed. Elastic-plastic and residual stress distributions are obtained from inner radius to outer radius, and they are presented in tables and figures. All radial stress components, σ_r, are compressive, and they are highest at the inner radius. All tangential stress components, σ_θ, are tensile, and they are highest where the plastic deformation begins. Magnitude of the tangential residual stresses is higher than those the radial residual stresses.

Key Words : Elastic-plastic stress analysis, Residual stresses, Aluminum disc

İÇ BASINCA MARUZ BİR ALUMİNYUM DİSK'İN ELASTİK-PLASTİK VE ARTIK GERİLME ANALİZİ

ÖZET

Bu makale, iç basınçlara maruz ince bir aliminyum diskte oluşan elastik- plastik gerilmelerin hesaplanması ile ilgilidir. Bu çalışmada, küçük plastik deformasyonlar için sınır şartlarını, elastik-plastik gerilmeri ve şekil değiştirmeleri sağlayan analitik bir çözüm öngörüldü. Von-Mises kriteri akma kriteri olarak kullanıldı ve malzeme plastik deformasyonda ideal plastik olarak kabul edildi. İç çaptan dış çapa kadar olan bölgede elastik-plastik ve artık gerilmeler hesaplandı. Sonuçlar tablolar ve grafikler halinde verildi. Bütün radial gerilme bileşenleri, σ_r, bası gerilmesidir ve bu gerilmeler disk'in iç çapında maksimum değerdedir. Bütün teğetsel gerilme bileşenleri, σ_θ, çeki gerilmesidir ve bu gerilmeler plastik deformasyonun başladığı yerde maksimum değerdedir. Teğetsel artık gerilmeler radial artık gerilmelerinkinden çok çok büyüktür.

Anahtar Kelimeler : Elastik-plastik gerilme analizi, Artık gerilmeler, Aliminyum disk

1. INTRODUCTION

The elastic-plastic discs subjected to internal pressure are encountered in many mechanical and structural applications. For example, a disc may be under internal pressure due to shrink fit on a shaft. The behavior of an elastic-plastic annular disk of linear hardening, subjected to external pressure, was first investigated analytically by Gamer (1986, 1987). Güven (1988; 1992) investigated an elastic-plastic annular disk of variable thickness under external pressure using by analytical and numerical methods.

Beside the elastic-plastic stresses, residual stresses are particularly important because they may be lead to premature failure. Prediction and measurement of residual stresses are important in relation to production, design and performance of any part of engineering materials and machine components. Jahed and Shirazi (2001) studied loading and unloading of a thermoplastic disc. They calculated loading and residual stresses, and associated strains and displacements for shrink fitted rotating disc at elevated temperatures by using variable material properties (VMP) and numerical method.

In this study, elastic-plastic stress analysis is carried out on an aluminum disc under internal pressures. To obtain the plastic stress components, an analytical method developed by Bektas and Sayman (2001) is used. Residual stresses are obtained by using elastic unloading technique. Results, elastic-plastic and residual stress components, are presented in Tables and Figures.

2. ELASTIC SOLUTION

The aluminum disc is assumed very thin, as seen in Figure 1, so that in *z* direction, stress and strain components are neglected, and shear stress is zero, $\tau_{r\theta} = 0$. The tangential and radial stresses under internal pressure, p, can be written respectively as, Timoshenko and Goodier (1970);

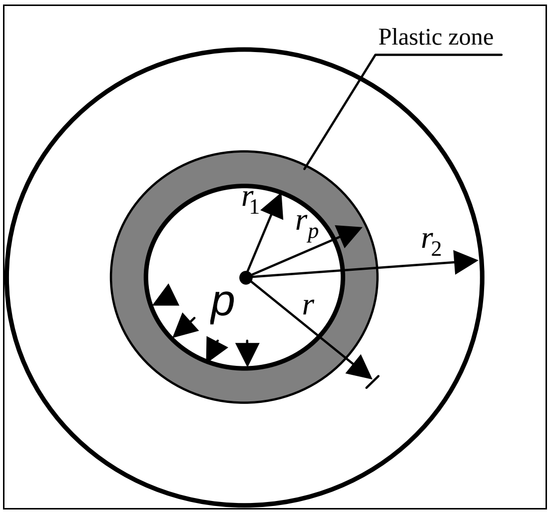

Figure 1. An aluminum disc with a hole under internal pressure, p, has a unit thickness, inner radius, $r_1 = a = 50$ mm, outer radius, $r_2 = b = 100$ mm, radius at yield, r_p and radius at any point, r, and all its edges are free

$$\sigma_r = \frac{p}{1-c^2}\left(c^2 - \frac{c^2 b^2}{r^2}\right)$$
$$\sigma_\theta = \frac{p}{1-c^2}\left(c^2 + \frac{c^2 b^2}{r^2}\right) \quad (1)$$

Where *r* is the radius at any point on the disc, and *a* is inner and *b* is the outer radius of the disc, and *c* is the ratio of them, $c = a/b$. They can be written in symbols as;

$$\sigma_r = p\,c_{r1}$$
$$\sigma_\theta = p\,c_{t1} \quad (2)$$

These stresses are depending on the radius r beside the internal pressure. For a specific radius r, those stresses are substituted into the Von Mises yield Criterion in order to find the internal pressure which is causing yielding of the disc, Mendelson (1968).

The radial and tangential strains are also written as;

$$\varepsilon_r = \frac{1}{E}(\sigma_r - \nu\sigma_\theta) = \frac{1}{E}\sigma_r - \frac{\nu}{E}\sigma_\theta = a_1\sigma_r - b_1\sigma_\theta$$
$$\varepsilon_\theta = \frac{1}{E}(\sigma_\theta - \nu\sigma_r) = \frac{1}{E}\sigma_\theta - \frac{\nu}{E}\sigma_r = a_1\sigma_\theta - b_1\sigma_r \quad (3)$$

Substituting Eq. (2) into Eq.(3) gives,

$$\varepsilon_r = p(a_1 c_{r1} - b_1 c_{t1}) \quad \text{or}$$
$$\varepsilon_r = p\,d_1$$
and
$$\varepsilon_\theta = p(a_1 c_{t1} - b_1 c_{r1}) \quad \text{or} \quad (4)$$
$$\varepsilon_r = p\,d_2$$

3. PLASTIC SOLUTION

The plastic solution is carried out for small plastic deformations. The Von Mises theory is used as a yield criterion due to the same yield points in tension and compression for the aluminum disc. The yield condition according to this criterion can be written as, Mendelson (1968);

$$\sigma_{eq} = \bar{\sigma} = \frac{1}{\sqrt{2}}\sqrt{(\sigma_\theta - \sigma_r)^2 + (\sigma_r - \sigma_z)^2 + (\sigma_z - \sigma_\theta)^2} \quad (5)$$

Since disc is very thin, stress in *z* direction is assumed as zero, $\sigma_z = 0$. So that Eq. (5) yields to,

$$\sigma_{eq} = \bar{\sigma} = \sqrt{\sigma_\theta^{\,2} + \sigma_r^{\,2} - \sigma_\theta\ \sigma_r} \tag{6}$$

Where the equivalent stress $\bar{\sigma}$ for an elastic perfectly plastic material is given as;

$$\bar{\sigma} = \sigma_Y \tag{7}$$

Where σ_Y is elastic limit. The plastic strain increments are obtained by using the potential function, $f = \bar{\sigma} - \sigma_Y(\varepsilon_P)$, Owen and Hinton (1980);

$$\begin{Bmatrix} d\varepsilon_r^p \\ d\varepsilon_\theta^p \\ d\varepsilon_{r\theta}^p \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial f}{\partial \sigma_r} d\lambda \\ \dfrac{\partial f}{\partial \sigma_\theta} d\lambda \\ \dfrac{\partial f}{\partial \tau_{r\theta}} d\lambda \end{Bmatrix} \tag{8}$$

where $d\lambda$ is equal to the equivalent plastic strain increment $d\varepsilon_p$. The total strain increments are written as,

$$\begin{aligned} d\varepsilon_r^p &= a_1\, d\sigma_r + b_1\, d\sigma_\theta + \frac{2\sigma_r - \sigma_\theta}{2\sigma_Y}\, d\varepsilon_p \\ d\varepsilon_\theta^p &= a_1\, d\sigma_r + b_1\, d\sigma_\theta + \frac{2\sigma_\theta - \sigma_r}{2\sigma_Y}\, d\varepsilon_p \\ d\varepsilon_{r\theta}^p &= 0 \end{aligned} \tag{9}$$

From the boundary conditions, the tangential and radial strains, Eq. (4), depending on the internal pressure for $a < r < b$ are substituted into Eq. (9), gives,

$$\begin{aligned} d\varepsilon_r^p &= a_1\, d\sigma_r + b_1\, d\sigma_\theta + \frac{2\sigma_r - \sigma_\theta}{2\sigma_Y}\, d\varepsilon_p - d_1\, dp = 0 \\ d\varepsilon_\theta^p &= a_1\, d\sigma_r + b_1\, d\sigma_\theta + \frac{2\sigma_\theta - \sigma_r}{2\sigma_Y}\, d\varepsilon_p - d_2\, dp = 0 \end{aligned} \tag{10}$$

The solution of these equations with the yield condition may be impossible. They can be written in a short and differential form as,

$$\begin{aligned} \Delta\varepsilon_r^p &= a_1\Delta\sigma_r + b_1\Delta\sigma_\theta + c_1\Delta\varepsilon_p - d_1\Delta p = 0 \\ \Delta\varepsilon_\theta^p &= a_1\Delta\sigma_r + b_1\Delta\sigma_\theta + c_2\Delta\varepsilon_p - d_2\Delta p = 0 \end{aligned} \tag{11}$$

Also Eq. (6) can be written in the differential form as;

$$(2\sigma_r - \sigma_\theta)\Delta\sigma_r + (2\sigma_\theta - \sigma_r)\Delta\sigma_\theta = 0 \tag{12}$$

and it is written in symbols

$$a_2\Delta\sigma_r + b_2\Delta\sigma_\theta = 0$$

$\Delta\sigma_r$, $\Delta\sigma_\theta$ and $\Delta\varepsilon_p$ are obtained from Eq. (11) and Eq. (12), and they are written as,

$$\begin{aligned} \Delta\sigma_r &= \frac{u_4 - u_2}{u_1 - u_3}\Delta p \\ \Delta\sigma_\theta &= \frac{u_4 - u_2}{u_5 - u_6}\Delta p \\ \Delta\varepsilon_p &= u_6\, \Delta\sigma_\theta + u_4\, \Delta p \end{aligned} \tag{13}$$

Where u_1, u_2, u_3, u_4, u_5 and u_6 are given in appendix.

If Δp is chosen at a small value, $\Delta\sigma_1$, $\Delta\sigma_2$ and $\Delta\varepsilon_p$ can be obtained more accurately. Stress components, σ_r, σ_θ and equivalent plastic strain ε_p can be found per step as,

$$\begin{aligned} (\sigma_r)_i &= (\sigma_r)_{i-1} + (\Delta\sigma_r)_i \\ (\sigma_\theta)_i &= (\sigma_\theta)_{i-1} + (\Delta\sigma_\theta)_i \\ (\varepsilon_p)_i &= (\varepsilon_p)_{i-1} + (\Delta\varepsilon_p)_i \end{aligned} \tag{14}$$

Where *i* is the number of the step.

4. RESIDUAL STRESSES

Residual stresses are obtained by superimposing elastic unloading stress distributions on elastic-plastic stress distributions. To obtain the elastic unloading stresses, the autofrettage pressure P_A required for yielding to any radius r_p is given by Hearn (1989), and it is written according to Von-Mises;

$$P_A = \sigma_Y \left[\frac{N^2 - m^2}{N^2\sqrt{3}} + \ln m \right] \tag{15}$$

Where $N = \dfrac{b}{a}$, $m = \dfrac{r_p}{a}$.

Elastic unloading stresses are obtained as;

$$(\sigma_r)_{unld} = P_A c_{r1}$$
$$(\sigma_\theta)_{unld} = P_A c_{t1} \tag{16}$$

The difference between elastic-plastic and elastic unloading stresses results in the residual stress components.

$$(\sigma_r)_r = \sigma_r - (\sigma_r)_{unld}$$
$$(\sigma_\theta)_r = \sigma_\theta - (\sigma_\theta)_{unld} \tag{17}$$

Those residual stresses are obtained numerically for any radius *r* on the disc.

5. A SAMPLE AND DISCUSSION

In this study an aluminum disc is used, as seen in Figure 1, and mechanical properties of the disc are given in Table 1. All the results are obtained by using the numerical solution. During the solution of the problem, pressure increment, Δp is chosen to be at a small value as 0.01, and the stress-strain components σ_r, σ_θ, ε_r, ε_θ, ε_p and residual stress components, $(\sigma_r)_r$, $(\sigma_\theta)_r$ are obtained for r = 50, 60, 70, 80, 90 and 100 mm. Results for the stress components at the beginning of plastic yielding at r = a and r = b are shown in Table 2.

Table 1. Mechanical properties and yield strengths of aluminum disc.

Modulus of Elasticity E (MPa)	Yield Strength σ_Y (MPa)	Poisson's Ratio ν
69000	230.0	0.3

Table 2. Pressures and Stress-Strain Components at the Beginning of Plastic Yielding

Pressure p (MPa)	Radius r (mm)	σ_r (MPa)	σ_θ (MPa)	$\varepsilon_r .(10^{-3})$	$\varepsilon_\theta .(10^{-3})$
-98.57	50	-98.57	164.29	-2.14	2.81
-345	100	0.00	230.00	-1.00	3.33

The elastic-plastic stresses and strains at the radius r for the applied pressure p are given in Table 3. As seen in this Table, all the stress and strain components in plastic zone are underlined. All radial stress components, σ_r are compressive, and they are highest at inner radius r = a, and zero at outer radius r = b. All tangential stress components, σ_θ are tensile, and they are highest where the plastic deformation begins, and minimum at outer radius r = b.

All radial residual stress components, $(\sigma_r)_r$ and tangential residual stress components, $(\sigma_\theta)_r$ are presented in Table 3. They are parabolic and changed from inner to outer radius. Radial ε_r, tangential ε_θ and equivalent ε_p plastic strains are also presented in Table 3. They are highest at inner radius *r = a* and minimum at outer radius *r* = b.

In Figure 2, the distribution of tangential stress components, σ_θ, are shown depending on the applied internal pressures, *p* and radius *r*. It changes nonlinearly from inner radius to outer radius. In this figure, above doted line shows plastic stress zone, and this line increases gradually since plastic zone expands. When the plastic zone expands, tangential stress σ_θ at yield point gets higher value whereas radial stress σ_r decreases.

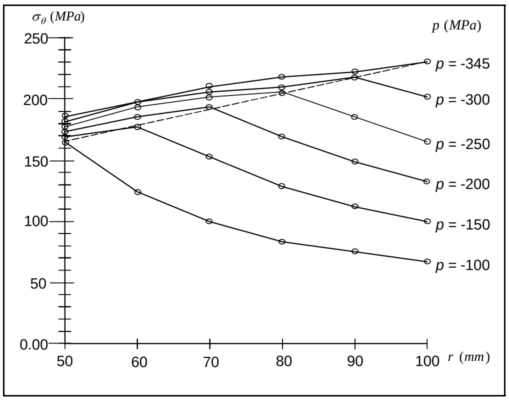

Figure 2. The distribution of tangential stress components, σ_θ at radius *r* under different internal pressures, p.

In Figure 3, the distribution of radial stress components, σ_r are shown for each applied internal pressures *p* and radius *r*. In this figure, above doted line shows plastic stress zone, but this zone increases in negative sign whereas its absolute value decreases, and this doted line decreases gradually. When the plastic zone expands, radial stress σ_r at yield point gets lower value whereas tangential stress σ_θ gets higher.

In Figure 4 and 5, tangential and radial residual stress distributions, $(\sigma_\theta)_r$, $(\sigma_r)_r$ are shown depending on the applied internal pressures *p* and radius *r*. Magnitude of the residual stresses increases while applied pressure is getting higher. Magnitude of the tangential residual stresses $(\sigma_\theta)_r$ is higher than the radial residual stresses $(\sigma_r)_r$. In Figure 4, the magnitude of the residual stress components $(\sigma_\theta)_r$ change from negative level to positive, and they are minimum at radius r = *60*. Figure 5, the magnitude of the residual stress components $(\sigma_r)_r$ change from negative level to positive, and they are zero at radius *r* = b.

Table 3. Elastic-Plastic Stress and Strain Components at the Disc Under Internal Pressures (Underlined Results are the Plastic Stress and Strain Components)

Pressure p (MPa)	Radius *r* (mm)	Radial and Tangential Stresses		Residual Stresses		Radial, Tangential and Equivalent Plastic Strains		
		σ_r (MPa)	σ_θ (MPa)	$(\sigma_r)_r$ (MPa)	$(\sigma_\theta)_r$ (MPa)	$\varepsilon_r .(10^{-3})$	$\varepsilon_\theta .(10^{-3})$	$\varepsilon_p .(10^{-3})$
-100	50	-98.38	164.45	-1.21	1.54	-2.14	2.81	0.04
	60	-59.26	125.93	0.24	0.51	-1.40	2.08	0.00
	70	-34.69	101.36	0.14	0.41	-0.94	1.61	0.00
	80	-18.75	85.42	0.08	0.35	-0.64	1.31	0.00
	90	-7.82	74.49	0.03	0.30	-0.43	1.11	0.00
	100	0.00	66.67	0.00	0.27	-0.28	0.96	0.00
-150	50	-92.59	169.28	-38.63	-49.43	-2.07	2.85	1.52
	60	-81.91	177.83	4.15	12.59	-1.96	2.93	0.20
	70	-52.04	152.04	6.51	19.03	-1.41	2.42	0.00
	80	-28.13	128.13	3.52	16.04	-0.96	1.97	0.00
	90	-11.73	111.73	1.47	13.99	-0.65	1.67	0.00
	100	0.00	100.00	0.00	12.52	-0.43	1.44	0.00
-200	50	-88.35	172.73	-59.47	-73.64	-2.03	2.88	3.00
	60	-75.94	182.43	-11.66	-3.72	-1.89	2.97	1.24
	70	-63.49	191.59	-12.20	41.76	-1.75	3.05	0.19
	80	-37.50	170.83	-9.78	44.57	-1.28	2.63	0.00
	90	-15.64	148.97	-4.08	38.87	-0.87	2.27	0.00
	100	0.00	133.33	0.00	34.79	-0.57	1.93	0.00
-250	50	-85.25	175.20	-72.03	-86.94	-1.99	2.90	4.48
	60	-71.16	186.01	-22.05	-12.05	-1.84	3.00	2.29
	70	-57.26	195.96	2.69	36.54	-1.68	3.08	0.98
	80	-43.32	205.26	13.83	70.81	-1.52	3.16	0.13
	90	-19.55	186.21	7.25	69.06	-1.09	2.78	0.00
	100	0.00	166.67	0.00	61.81	-0.724	2.41	0.00
-300	50	-83.00	176.98	-77.56	-90.62	-1.97	2.92	5.97
	60	-67.34	188.82	-27.81	-13.37	-1.79	3.02	3.35
	70	-52.01	199.54	-3.70	36.80	-1.62	3.11	1.78
	80	-36.87	209.34	6.77	72.19	-1.44	3.19	0.77
	90	-21.77	218.34	9.22	98.75	-1.26	3.25	0.08
	100	0.00	200.00	0.00	92.96	-0.86	1.89	0.00
-345	50	-81.51	178.14	-78.24	-88.11	-1.95	2.93	7.31
	60	-64.57	190.81	-30.10	-10.36	-1.76	3.04	4.30
	70	-47.99	202.22	-7.44	40.29	-1.57	3.13	2.50
	80	-31.78	212.46	1.83	76.00	-1.38	3.21	1.34
	90	-15.81	221.69	3.32	102.69	-1.19	3.28	0.56
	100	0.00	230.00	0.00	123.50	-1.00	3.33	0.00

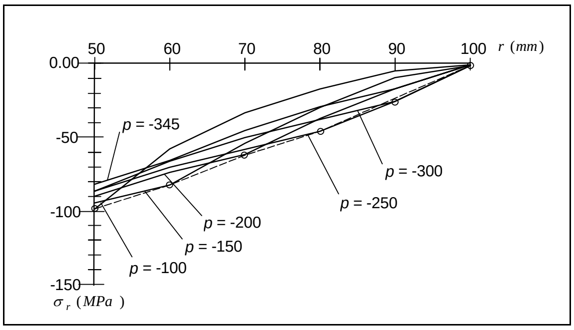

Figure 3. The distribution of radial stress components, σ_r at radius *r* under different internal pressures, p.

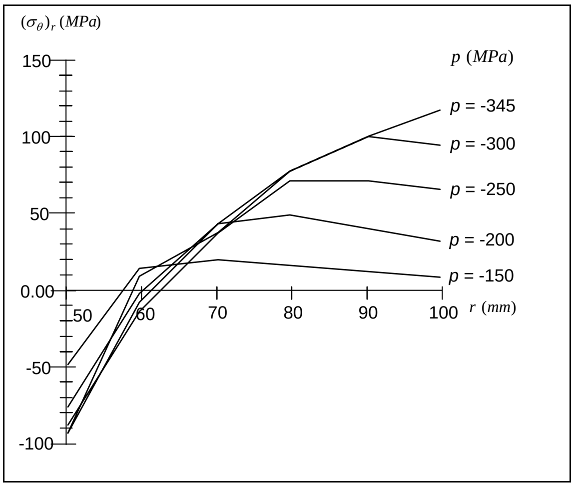

Figure 4. The distribution of tangential residual stress components, $(\sigma_\theta)_r$ at radius *r* under different internal pressures, p.

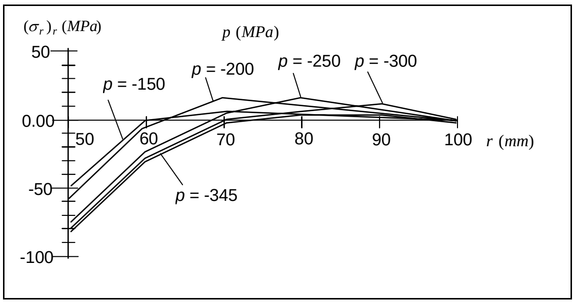

Figure 5. The distribution of radial residual stress components, $(\sigma_r)_r$ at radius r under different internal pressures, *p*.

6. CONCLUSIONS

1. All radial stress components, σ_r, are compressive, and they are highest at inner radius, *r = a*, and zero at outer radius, *r = b*.
2. All tangential stress components, σ_θ, are tensile, and they are highest at inner radius *r = a*, and minimum at outer radius *r = b*.
3. Plastic zone begins at *r = a* and spreads through the outer radius as applied pressure is increased.
4. After yielding, plastic stress components increase gradually since plastic region expands.
5. Magnitude of the tangential residual stresses is higher than those radial residual stresses.

7. APPENDICS

$$u_1 = \frac{b_1\,a_2}{b_2\,c_1} - \frac{a_1}{c_1},\; u_2 = \frac{d_1}{c_1},\; u_3 = \frac{b_1\,a_2}{b_2\,c_2} - \frac{a_1}{c_2},$$

$$u_4 = \frac{d_2}{c_2},\; u_5 = \frac{a_1\,b_2}{a_2\,c_1} - \frac{b_1}{c_1},\; u_6 = \frac{a_1\,b_2}{a_2\,c_2} - \frac{b_1}{c_2}$$

8. REFERENCES

Bektaş, N. B. and Sayman, O. 2001. Elasto-Plastic Stress Analysis in Simply Supported Thermoplastic Laminated Plates Under Thermal Loads, Composites Science and Technology, V.61, pp. 1695-1701, 2001.

Gamer, U. 1986. The Rotating Elastic-Plastic Shrink Fit With Hardening. Acta Mech. ; 61: 15-27.

Gamer, U. 1987. The Shrink Fit With Elastic Plastic Hub Exhibiting Constant Yield Stress Followed by Hardening. Int. J. Solids Struct. ; 23: 1219-24.

Güven, U. 1988. Stress Distribution in a Linear Hardening Annular Disk of Variable Thickness Subjected to External Pressure. Int. J. Mech. Sci. Vol. 40, No: 6, pp. 585-601.

Güven, U. 1992. Elastic-plastic Annular Disk With Variable Thickness Subjected to External Pressure. Acta Mechanica, 92: 29-34.

Hearn, E.J. 1989. Mechanics of Materials, Pergamon Press, Oxford

Jahed H. and Shirazi R. 2001. Loading and Unloading Behaviour of a Thermoplastic Disc, Int. J. of Pressure Vessels And Piping, 78; 637-645.

Mendelson A. 1968. Plasticity: Theory and Application, The Macmillan Company.

Owen, D.R.J. and Hinton, E. 1980. Finite Elements in Plasticity, Pineridge Press Limited, Swansca.

Timoshenko S. P. and Goodier J.N. 1970. Theory of Elasticity, McGraw-Hill.